The ellipse in astronomy

Dutch

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Introduction

In 1609 the mathematician and astronomer Johannes Kepler (1571-1630) published his famous work Astronomia Nova seu Physica_coelestis. In it he unfolded his calculations of the planetary movements and the laws related to them.

In order to arrive at these insights, he could base himself on a large amount of observational material, which had been collected on the island of Hven by his teacher and patron Tycho Brahe (1546-1601). Brahe had built two very well-equipped observatories on the island, thanks to the generous financial resources made available to him by the Danish king. In addition, he had some of the most advanced measuring instruments (for his time) at his disposal, with which the position of stars and planets could be accurately determined. The measurements were made with the naked eye, because the telescope had not yet been invented.

In 1599 Tycho Brahe left for Prague, where he came into contact with Johannes Kepler. Shortly before Brahe's death, Kepler received his data, which he used to calculate an accurate planetary orbit. It took him 5 years to determine his first exact planetary orbit, that of Mars. First, he discovered the “law of equal areas” (Kepler's second law, see below). He then discovered that the orbits of Mars and the other planets were not an epicycle, as Claudius Ptolemy (c. 90-160) and Nicolaus Copernicus (1473-1543) had claimed, but an ellipse.
Kepler discovered his third law (see below) some 10 years later and is set forth in his book Harmonices Mundi (1619).

Kepler’s three laws can be formulated as follows:

1. The planets describe elliptical orbits around the sun, with the sun in one of the foci.

2. The motion of the planets is such that the radius vector (the line connecting the sun and the planet) covers equal areas in equal time intervals.

3. The ratio of the square of the time of revolution (orbital period) and the cube of the semimajor axis of the ellipse is constant for all planets.

The laws empirically found by Kepler could be explained some 75 years later by Isaac Newton (1643-1727) using the concept of gravity, as described in his major work Philosophiae Naturalis Principia Mathematica (1687).
In addition, Newton is the discoverer of differential and integral calculus, the necessary tool to mathematically describe planetary motions.

The equation of the ellipse in polar coordinates

According to Kepler's first law, the planets revolve around the sun in elliptical orbits, with the sun at one of the foci. Figure 1 shows such an elliptical orbit, where the point P represents the position of the planet at a particular time. The sun is at the right focal point S of the ellipse, which is also the origin of the x- and y-axis coordinate system.

The intersections of the ellipse with the x-axis are formed by the perihelion Π, at which the planet is closest to the sun, and the aphelion A, the position of the planet farthest from the sun. The point O is the center of the ellipse.

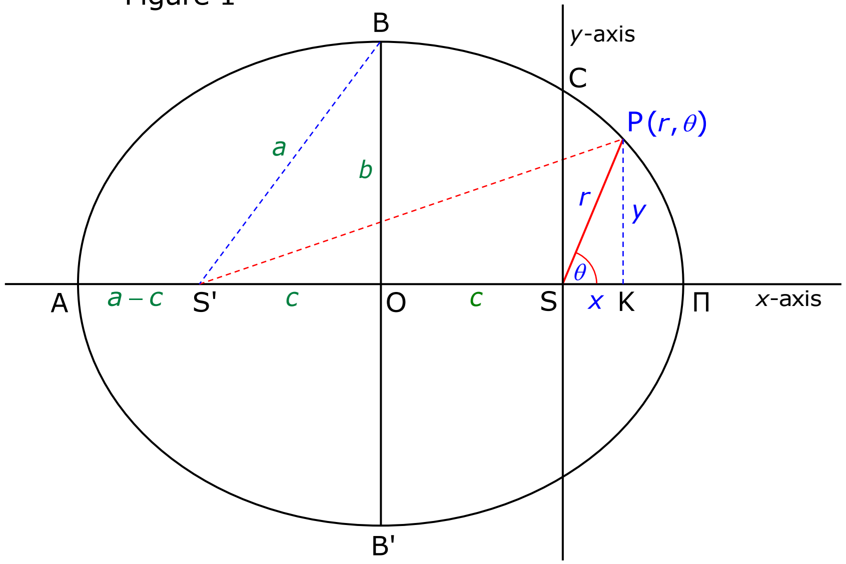
The position P of the celestial body is described by the coordinates r en θ, which are defined as follows:

- the radius vector r with $r = \text{PS}$ the distance from the celestial body to the sun;
- the true anomaly θ with $\theta = \angle \Pi\text{SP}$ the angle of the radius vector with the x-axis, measured from the perihelion.

To characterize the ellipse, the following parameters are important:

- the semi-major axis a with $a = \text{O}\Pi = \text{OA}$
- the semi-minor axis b with $b = \text{OB} = \text{OB'}$
- the focal distance c with $c = \text{OS} = \text{OS'}$
- the eccentricity e with $e = \dfrac{c}{a}$ $\quad (0 \le e < 1)$ (1)
- the ellipse parameter p with $p = a(1 - e^2)$ $\quad (p \le a)$ (2)

Figure 1



For each point of the ellipse holds by definition: $\text{PS} + \text{PS'} = 2a = \text{constant}$ (3)

From this also follows: $\text{BS} + \text{BS'} = 2a \;\rightarrow\; \text{BS} = \text{BS'} = a \;\rightarrow\; a^2 = b^2 + c^2$ (4)

Furthermore, the following applies:

$$x = \text{KS} = r\cos\theta \qquad y = \text{KP} = r\sin\theta \qquad r = \text{PS} = \sqrt{x^2 + y^2} \tag{5}$$

The equation of the ellipse can be derived by applying (1), (2) and (3):

$$\text{PS} = r \quad \text{and} \quad \text{PS'} = \sqrt{\text{S'K}^2 + \text{KP}^2} = \sqrt{(2c + x)^2 + y^2}$$

$$\text{PS} + \text{PS'} = 2a = r + \sqrt{(2c + r\cos\theta)^2 + (r\sin\theta)^2} = r + \sqrt{4c^2 + 4rc\cos\theta + r^2}$$

$$(2a - r)^2 = 4c^2 + 4rc\cos\theta + r^2 \quad \rightarrow \quad a^2 - ar = c^2 + rc\cos\theta$$

$$a^2 - c^2 = b^2 = r(a + c\cos\theta) \quad \rightarrow \quad \frac{a^2 - c^2}{a} = r(1 + \frac{c}{a}\cos\theta) \quad \rightarrow \quad a(1 - \frac{c^2}{a^2}) = a(1 - e^2) = p = r(1 + e\cos\theta)$$

From this follows the equation of the ellipse in polar coordinates:

$$r = \frac{p}{1 + e\cos\theta} \tag{6}$$

The equation (6) can be solved for some characteristic angles , using definition (2):

- $\theta = 0 \quad \rightarrow \quad r = \frac{p}{1 + e\cos 0} = \frac{p}{1+e} = \frac{a(1-e^2)}{1+e} = a(1-e) = \Pi S$ distance perihelion-sun (7)
- $\theta = \frac{1}{2}\pi \quad \rightarrow \quad r = \frac{p}{1 + e\cos\frac{1}{2}\pi} = p = SC$ geometric meaning of the ellipse parameter p (8)
- $\theta = \pi \quad \rightarrow \quad r = \frac{p}{1 + e\cos\pi} = \frac{p}{1-e} = \frac{a(1-e^2)}{1-e} = a(1+e) = AS$ distance aphelion-sun (9)

Furthermore, when applying the formulas (1), (2), (4) and (7):

$$a = \frac{p}{1-e^2} \tag{10}$$

$$c = OS = O\Pi - \Pi S = a - a(1-e) = ae \tag{11}$$

$$b^2 = a^2 - c^2 = a^2 - (ae)^2 = a^2(1-e^2) = ap \quad \rightarrow \quad b = a\sqrt{1-e^2} = \sqrt{ap} \tag{12}$$

$$\frac{b}{a} = \sqrt{1-e^2} \tag{13}$$

$$b = a\sqrt{1-e^2} = \frac{p}{1-e^2}\sqrt{1-e^2} = \frac{p}{\sqrt{1-e^2}} \tag{14}$$

$$p = \frac{b^2}{a} \tag{15}$$

$$e = \sqrt{\frac{c^2}{a^2}} = \sqrt{\frac{a^2-b^2}{a^2}} = \sqrt{1-\frac{b^2}{a^2}} \tag{16}$$

In case the eccentricity becomes 0, then $a = b = p$ and the ellipse turns into a circle.

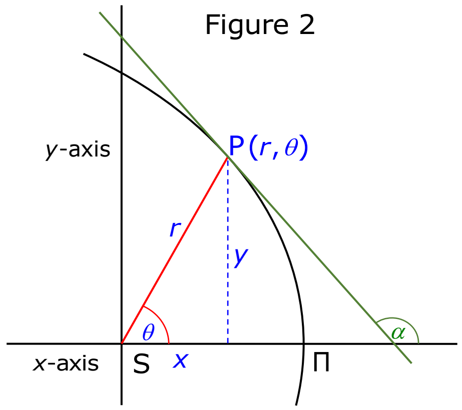
When the eccentricity approaches the value 1, the shape of the orbit approaches a parabola.

The tangent to the ellipse

At very small displacements in its orbit around the sun, the motion of a planet is approximately directed along the tangent to the ellipse. In the following it will be seen that the direction of this tangent line is important for the speed.

In figure 2 such a tangent line is drawn, tangent to the ellipse in point P. This makes an angle α with the x-axis. It will now be deduced how this angle can be expressed in the true anomaly θ.

In rectangular coordinates, the slope of the tangent line is given by: $\tan\alpha = \dfrac{dy}{dx}$



Differentiating the formulas (5) $x = r\cos\theta$ and $y = r\sin\theta$ gives:

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta \quad \text{and} \quad dy = \sin\theta\, dr + r\cos\theta\, d\theta \tag{17}$$

So the slope of the tangent line is:

$$\frac{dy}{dx} = \frac{\sin\theta\, dr + r\cos\theta\, d\theta}{\cos\theta\, dr - r\sin\theta\, d\theta} = \frac{\sin\theta\, dr - \dfrac{r\cos\theta}{\sin\theta}\, d(\cos\theta)}{\cos\theta\, dr + r\, d(\cos\theta)} \tag{18}$$

Differentiating the orbit equation (6) $r = \dfrac{p}{1 + e\cos\theta}$ gives the following result:

$$dr = \frac{pe\sin\theta}{(1+e\cos\theta)^2}\, d\theta = \left(\frac{p}{1+e\cos\theta}\right)^2 \frac{e}{p}\sin\theta\, d\theta = \frac{e}{p} r^2 \sin\theta\, d\theta = -\frac{e}{p} r^2 d(\cos\theta) \tag{19}$$

Plugging in the right-hand expression of dr in equation (18) and then multiplying the numerator and denominator by $\dfrac{p}{r^2}\sin\theta$ gives:

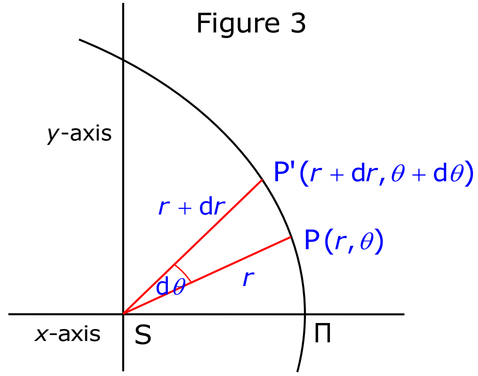
$$\frac{dy}{dx} = \frac{-e\sin^2\theta - \dfrac{p}{r}\cos\theta}{-e\cos\theta\sin\theta + \dfrac{p}{r}\sin\theta} = -\frac{e(1-\cos^2\theta) + \dfrac{p}{r}\cos\theta}{\sin\theta\left(-e\cos\theta + \dfrac{p}{r}\right)} = -\frac{e + \cos\theta\left(-e\cos\theta + \dfrac{p}{r}\right)}{\sin\theta\left(-e\cos\theta + \dfrac{p}{r}\right)}$$

From the orbit equation (6) it follows $-e\cos\theta + \dfrac{p}{r} = 1$, which shows the relationship between α and θ immediately:

$$\tan\alpha = \frac{dy}{dx} = -\frac{e + \cos\theta}{\sin\theta} \tag{20}$$

The second law and the orbital motion

According to Kepler's second law (the law of equal areas), the radius vector (the connecting line sun-planet) covers equal areas in equal time intervals. This law can be used to describe the motion of the planet as a function of time.



To do this, consider figure 3, where a small portion of the track is shown. The planet moves in it from position P to position P'. In a very short period of time dt the radius vector increases thereby in length from r to $r+\mathrm{d}r$, while the true anomaly increases with the small angle $\mathrm{d}\theta$. The radius vector here covers the area of the ellipse sector PSP'.

The sector area $\mathrm{d}A$ can be approximated by a triangle, because the arc PP' can be considered a straight line. Approximately then:

$$\mathrm{d}A = \frac{1}{2} r(r+\mathrm{d}r)\sin(\mathrm{d}\theta) \cong \frac{1}{2} r^2 \mathrm{d}\theta$$

According to Kepler's second law, this area is proportional to the elapsed time dt (with proportionality constant h), so that in the infinitesimal case the following holds:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2} r^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} = h = \text{constant} \qquad \text{(Kepler's second law)} \qquad (21)$$

This law can also be formulated in rectangular coordinates, if the formulas (5) are used:

$$x = r\cos\theta \quad \text{en} \quad y = r\sin\theta \quad \rightarrow \quad \tan\theta = \frac{y}{x}$$

Differentiating over time from the latter formula then yields:

$$\frac{\mathrm{d}(\tan\theta)}{dt} = \frac{1}{\cos^2\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{x^2}\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right) \rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\cos^2\theta}{x^2}\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{1}{r^2}\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right) \rightarrow r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}$$

Substituting the latter in equation (21) then gives:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right) = h = \text{constant} \qquad (22)$$

The differential equation (21) makes it possible to determine the constant, if the orbital period of the planet is known.
The orbital period is the time it takes for the planet to traverse the entire ellipse orbit so that the radius vector covers the entire area of the ellipse. Consequently, the true anomaly ranges from $\theta = 0$ to $\theta = 2\pi$.

Using formula (12) the area of the ellipse is given by (see also the derivation in appendix A):

$$A(\text{ellipse}) = \pi\, ab = \pi\, a\sqrt{ap} = \pi\, a^{3/2}\sqrt{p}$$

Integrating the equation (21) then gives:

$$\int_0^{2\pi} \frac{1}{2} r^2 \mathrm{d}\theta = h\int_0^T \mathrm{d}t \qquad \text{with} \qquad \frac{1}{2}\int_0^{2\pi} r^2 \mathrm{d}\theta = A(\text{ellips}) = \pi\, a^{3/2}\sqrt{p} \qquad \text{and} \qquad h\int_0^T \mathrm{d}t = ht\,\Big|_0^T = hT$$

from which follows: $h = \dfrac{\pi\, ab}{T} = \dfrac{\pi\, a^{3/2}\sqrt{p}}{T}$ (23)

Assuming that the factor $\dfrac{h^2}{p}$ is constant and identical for all celestial bodies moving in elliptical orbits around the sun, the formula (23) is in fact a mathematical formulation of Kepler's third law (see also pages 12 and 13):

> The ratio of the square of the period of revolution T and the cube of the semimajor axis a is constant for all planets.

After all: $\dfrac{1}{\pi^2}\dfrac{h^2}{p} = \dfrac{a^3}{T^2} = \text{constant}$ (Kepler's third law) (24)

To calculate the true anomaly θ of the planet at any given time t, the mathematical formulation (21) of Kepler's second law must be integrated. The equation of the ellipse (6) and the constant h (23) can be used for this:

$$A(t) = \int_0^{\theta} \frac{1}{2} r^2 \mathrm{d}\theta = \frac{1}{2}\int_0^{\theta} \frac{p^2 \mathrm{d}\theta}{(1+e\cos\theta)^2} = h\int_{t_0}^{t} \mathrm{d}t = \frac{\pi\, ab}{T}(t-t_0) \quad \rightarrow \quad A(t) = \frac{1}{2}p^2\int_0^{\theta} \frac{\mathrm{d}\theta}{(1+e\cos\theta)^2} = \frac{\pi\, ab}{T}(t-t_0) \qquad (25)$$

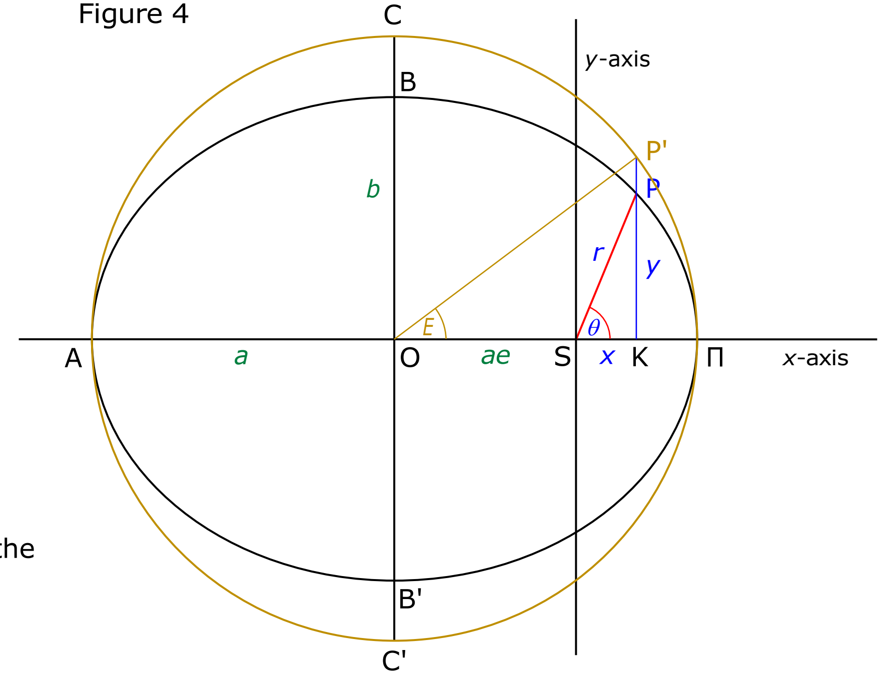
$A(t)$ is here the area covered by the radius vector at the true anomaly θ at time t, calculated from $\theta = 0$ at time t_0.
The value t_0 indicates the time when the planet crosses perihelion (where the true anomaly $\theta = 0$).

The integral (25) is not easy to solve. To this end, an auxiliary parameter E, the eccentric anomaly, must first be introduced.

The eccentric anomaly

Following Kepler's example, the ellipse is provided with a circumscribed circle, so that its radius is equal to the semimajor axis a.
The point P' is the intersection of this circle with the perpendicular through P to the x-axis.
When the planet at point P moves along the ellipse, this point P' moves about the circle. The angle between the segment OP' and the x-axis becomes the so called eccentric anomaly, thus $E = \angle \Pi OP'$ (see figure 4).
The perihelion and aphelion are the only two points at which $E = \theta$.

Figure 4



Applying the formulas (5) and (11), a relationship between the three variables E, θ and r can be easily derived from the figure:

$$x = r\cos\theta = \text{OK} - \text{OS} = a\cos E - ae = a(\cos E - e) \tag{26}$$

Another relationship between the variables E and r can be found as follows:

According to definition (2) and formula (6): $r + er\cos\theta = p = a(1-e^2)$

Multiplying relation (26) by e gives: $er\cos\theta = ae\cos E - ae^2$

Subtracting the two results above gives:

$$r = a - ae\cos E = a(1 - e\cos E) \tag{27}$$

A third relationship can be found via the well-known property of the ellipse, that the ratio of the y-coordinate of points P on the ellipse and the y-coordinate of the corresponding points P' on the circumscribed circle according to (13) is equal to:

$$\frac{\text{KP}}{\text{KP'}} = \frac{\text{OB}}{\text{OC}} = \frac{b}{a} = \sqrt{1-e^2}$$

Because $\text{KP} = y = r\sin\theta$ and $\text{KP'} = a\sin E$ therefore holds: $y = r\sin\theta = b\sin E = a\sqrt{1-e^2}\,\sin E$ (28)

This equation can also be derived from only equations (26) and (27). See appendix B for this.

There is also a direct relation between the true anomaly θ and the eccentric anomaly E.

Subtracting and adding the equations (26) and (27) gives, respectively:

$$r - r\cos\theta = a - ae\cos E - a\cos E + ae \quad\rightarrow\quad r(1-\cos\theta) = a(1+e)(1-\cos E)$$
$$r + r\cos\theta = a - ae\cos E + a\cos E - ae \quad\rightarrow\quad r(1+\cos\theta) = a(1-e)(1+\cos E)$$

By passing to the half angle via the trigonometric identities $1-\cos\theta = 2\sin^2\frac{1}{2}\theta$ and $1-\cos E = 2\sin^2\frac{1}{2}E$ resp. $1+\cos\theta = 2\cos^2\frac{1}{2}\theta$ and $1+\cos E = 2\cos^2\frac{1}{2}E$ and after taking square roots the following equations arise:

$$\sqrt{r}\,\sin\frac{1}{2}\theta = \sqrt{a(1+e)}\,\sin\frac{1}{2}E \quad \text{and} \quad \sqrt{r}\,\cos\frac{1}{2}\theta = \sqrt{a(1-e)}\,\cos\frac{1}{2}E \tag{29}$$

so by dividing the equations (29) it follows:

$$\tan\frac{1}{2}\theta = \sqrt{\frac{1+e}{1-e}}\,\tan\frac{1}{2}E \tag{30}$$

Kepler's equation

Using the eccentric anomaly E introduced above, equation (25) can be integrated.

Differentiating equation (27) $r = a(1-e\cos E)$ gives:

$$\mathrm{d}r = ae\sin E\,\mathrm{d}E \tag{31}$$

According to (19) differentiating the orbit equation gives:

$$\mathrm{d}r = \frac{e}{p}r^2\sin\theta\,\mathrm{d}\theta$$

Equating the last two results then yields:

$$\frac{p^2\mathrm{d}\theta}{(1+e\cos\theta)^2} = r^2\mathrm{d}\theta = ap\frac{\sin E}{\sin\theta}\mathrm{d}E = b^2\frac{\sin E}{\sin\theta}\mathrm{d}E$$

Equation (28) gives $\frac{\sin E}{\sin\theta} = \frac{r}{b}$, so that:

$$\frac{p^2\mathrm{d}\theta}{(1+e\cos\theta)^2} = br\,\mathrm{d}E$$

Substitution of r via equation (27) then leads to:

$$\frac{p^2\mathrm{d}\theta}{(1+e\cos\theta)^2} = ab(1-e\cos E)\,\mathrm{d}E$$

For the integral (25) can now be written:

$$A(t) = \frac{1}{2}p^2\int_0^\theta \frac{\mathrm{d}\theta}{(1+e\cos\theta)^2} = \frac{1}{2}\int_0^E ab(1-e\cos E)\,\mathrm{d}E = \frac{1}{2}ab(E-e\sin E)\Big|_0^E = \frac{1}{2}ab(E-e\sin E) = \frac{\pi ab}{T}(t-t_0)$$

The final result is Kepler's equation:

$$E - e\sin E = \frac{2\pi}{T}(t-t_0) = M \tag{32}$$

The factor $\frac{2\pi}{T}(t-t_0)$ is also called the mean anomaly M, so $M=\frac{2\pi}{T}(t-t_0)$ (33)

This corresponds to the angle which increases proportionally with time from 0 to 2π during one single revolution of the planet.
The mean anomaly can be understood as an angle of a fictitious celestial body, which at a constant speed describes a circular orbit with the same period as the planet.
In the perihelion Π and the aphelion A, all anomalies are equal, so $\theta_\Pi = E_\Pi = M_\Pi = 0$ respectively $\theta_\mathrm{A} = E_\mathrm{A} = M_\mathrm{A} = \pi$.

Kepler's equation is a transcendental equation, from which the eccentric anomaly can only be solved iteratively.
The following iteration formula is particularly suitable for small eccentricities:

$$E_n = M + e\sin E_{n-1} \tag{34}$$

in which $E_0 = M$ can be taken as the initial value. The calculation should be continued until the successively calculated eccentric anomalies equalize within the required accuracy: $E_n = E_{n-1}$.

At larger values of the eccentricity, formula (34) converges poorly, so it is better to use the following method:

$$E_n = E_{n-1} + \frac{M + e\sin E_{n-1} - E_{n-1}}{1 - e\cos E_{n-1}} \tag{35}$$

Here too, the starting value $E_0 = M$ can be taken. The latter formula is based on the Newton-Raphson method, a derivation of which can be found in appendix L.

Once the value of E is known, the rectangular coordinates x and y can be calculated using formulas (26) and (28).
The radius vector r and the true anomaly θ can be determined by the formulas (27) and (30).

The area $A(t)$ covered by the radius vector can now also be calculated according to formulas (25) and (32) via:

$$A(t) = h(t-t_0) = \frac{\pi ab}{T}(t-t_0) = \frac{\pi a^{3/2}\sqrt{p}}{T}(t-t_0) = \frac{1}{2}a^{3/2}\sqrt{p}\cdot(E - e\sin E) \tag{36}$$

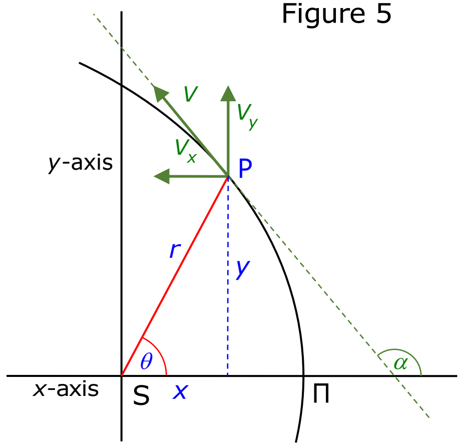
In Kepler's time differential and integral calculus had not yet been invented. Kepler has therefore found the equation (32) by geometric means. See appendix C for this, where a geometric derivation can be found.

The speed of the planet

As the planet traverses its elliptical orbit, the magnitude and direction of its speed are constantly changing, depending on its distance from the sun. The speed is directed along the tangent to the ellipse (see figure 5).

For the speed V holds (see also appendix D for a derivation):

$$V^2 = (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = (\frac{dr}{dt})^2 + r^2(\frac{d\theta}{dt})^2 \qquad (37)$$



According to formula (19) holds: $\frac{dr}{dt} = \frac{e}{p} r^2 \sin\theta \frac{d\theta}{dt}$ (38)

and according to formula (21): $\frac{1}{2} r^2 \frac{d\theta}{dt} = h \quad \rightarrow \quad r\frac{d\theta}{dt} = \frac{2h}{r}$ (39)

Combining both results and applying the orbit equation (6) then gives:

$$\frac{dr}{dt} = \frac{2h}{p} e \sin\theta \qquad \text{and} \qquad r\frac{d\theta}{dt} = \frac{2h}{r} = \frac{2h}{p}(1 + e\cos\theta) \qquad (40)$$

Plugging the formulas (40) into the speed formula (37) yields:

$$V^2 = (\frac{dr}{dt})^2 + r^2(\frac{d\theta}{dt})^2 = \frac{4h^2}{p^2} e^2 \sin^2\theta + \frac{4h^2}{p^2}(1 + e\cos\theta)^2 = \frac{4h^2}{p^2}(e^2\sin^2\theta + 1 + 2e\cos\theta + e^2\cos^2\theta) =$$

$$= \frac{4h^2}{p^2}(1 + 2e\cos\theta + e^2) = \frac{4h^2}{p}(2 \cdot \frac{1 + e\cos\theta}{p} - \frac{1 - e^2}{p}) = \frac{4h^2}{p}(\frac{2}{r} - \frac{1 - e^2}{p}) = \frac{4h^2}{p}(\frac{2}{r} - \frac{1}{a})$$

According to (24), the factor $\frac{4h^2}{p}$ equals: $\frac{4h^2}{p} = \frac{4\pi^2 a^3}{T^2}$

So the final formula for the speed becomes: $V = \frac{2\pi a^{3/2}}{T}\sqrt{\frac{2}{r} - \frac{1}{a}}$ (41)

In the perihelion, the speed is maximum and equal to: $V_{max} = \frac{2\pi a^{3/2}}{T}\sqrt{\frac{2}{a(1-e)} - \frac{1}{a}} = \frac{2\pi a}{T}\sqrt{\frac{1+e}{1-e}}$ (42)

In the aphelion, the velocity reaches its minimum value: $V_{min} = \frac{2\pi a^{3/2}}{T}\sqrt{\frac{2}{a(1+e)} - \frac{1}{a}} = \frac{2\pi a}{T}\sqrt{\frac{1-e}{1+e}}$ (43)

The speed components and the acceleration

The velocity components given in formula (37) can also be calculated separately, starting from formulas (17):

$$\frac{dx}{dt} = \cos\theta \frac{dr}{dt} - r\sin\theta \frac{d\theta}{dt} \quad \text{and} \quad \frac{dy}{dt} = \sin\theta \frac{dr}{dt} + r\cos\theta \frac{d\theta}{dt} \tag{44}$$

According to (40) the following applies: $r\frac{d\theta}{dt} = \frac{2h}{r} = \frac{2h}{p}(1 + e\cos\theta)$

Therefore, formula (38) can be written as: $\frac{dr}{dt} = \frac{e}{p} r^2 \sin\theta \frac{d\theta}{dt} = 2h\frac{e}{p}\sin\theta$

When applied via the orbit equation (6) using $er\cos\theta - p = -r$, substitution of the last formulas yields:

$$\frac{dx}{dt} = 2h\frac{e}{p}\sin\theta\cos\theta - \frac{2h}{r}\sin\theta = \frac{2h\sin\theta}{pr}(er\cos\theta - p) = \frac{2h\sin\theta}{pr}\cdot -r \quad \rightarrow \quad V_x = \frac{dx}{dt} = -\frac{2h}{p}\sin\theta \tag{45}$$

$$\frac{dy}{dt} = 2h\frac{e}{p}\sin^2\theta + \frac{2h}{p}\cos\theta\cdot(1 + e\cos\theta) = \frac{2h}{p}(e\sin^2\theta + \cos\theta + e\cos^2\theta) \quad \rightarrow \quad V_y = \frac{dy}{dt} = \frac{2h}{p}(e + \cos\theta) \tag{46}$$

The quotient of the velocity components gives $\frac{dy}{dx} = \tan\alpha = -\frac{e + \cos\theta}{\sin\theta}$, so that the direction of velocity coincides with that of the tangent, according to the formula (20).

In addition: $V^2 = (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = \frac{4h^2}{p^2}(\sin^2\theta + (e + \cos\theta)^2) = \frac{4h^2}{p^2}(\sin^2\theta + e^2 + 2e\cos\theta + \cos^2\theta) = \frac{4h^2}{p^2}(1 + 2e\cos\theta + e^2)$, which is entirely in agreement with the intermediate result for the derivation of the velocity formula (41).

Now that the velocity components are known, the acceleration of the planet can be calculated.

Differentiating the velocity components with respect to time, using formula (21), namely $\frac{d\theta}{dt} = \frac{2h}{r^2}$, gives:

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(-\frac{2h}{p}\sin\theta\right) = -\frac{2h}{p}\cos\theta\frac{d\theta}{dt} = -\frac{2h}{p}\cos\theta\cdot\frac{2h}{r^2} \quad\rightarrow\quad a_x = \frac{d^2x}{dt^2} = -\frac{4h^2}{p}\frac{\cos\theta}{r^2} \qquad (47)$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{2h}{p}(e+\cos\theta)\right) = -\frac{2h}{p}\sin\theta\frac{d\theta}{dt} = -\frac{2h}{p}\sin\theta\cdot\frac{2h}{r^2} \quad\rightarrow\quad a_y = \frac{d^2y}{dt^2} = -\frac{4h^2}{p}\frac{\sin\theta}{r^2} \qquad (48)$$

The acceleration a (not to be confused with the semimajor axis symbol of the ellipse) then becomes:

$$a^2 = a_x{}^2 + a_y{}^2 = \left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 = \frac{16h^4}{p^2}\frac{\cos^2\theta + \sin^2\theta}{r^4} = \frac{16h^4}{p^2}\frac{1}{r^4} \quad\rightarrow\quad a = -\frac{4h^2}{p}\frac{1}{r^2} \qquad (49)$$

This shows that the magnitude of the acceleration is inversely proportional to the square of the distance from the sun. This is the basis of Newton's theory of gravitation.

The direction of the acceleration is determined by the quotient of the horizontal and vertical acceleration components.

For this quotient holds $\frac{a_y}{a_x} = \frac{-\sin\theta}{-\cos\theta} = \tan\theta$, so the acceleration is directed according to the radius vector (the connecting line planet-sun).

From the fact that both a_x and a_y (in the first quadrant) are negative, it can be inferred that the acceleration is directed towards the sun.

Kepler's third law and the gravitational constant

The acceleration according to formula (49) indicates a central attractive force exerted by the sun. It's just unlikely, that the acceleration would still depend on the shape of the orbit, as the ellipse parameter p in this formula suggests.

According to formula (24) Kepler's third law shows, that the factor $\frac{4h^2}{p}$ is a constant for all planets in the solar system, and a characteristic factor for the sun. It follows that the proportionality factor h in Kepler's second law is inversely proportional to the square root of the parameter p.

The gravitational pull of the sun on a planet with mass m can be written (according to Newton) as:

$$F = m\,a = \frac{4h^2}{p}\frac{m}{r^2} \qquad (50)$$

The table below gives the factor $\frac{4h^2}{p}$ for 8 planets, 1 dwarf planet and 1 asteroid in our solar system (source: NASA 2016)

Object	a (AE)	e	T (days)	$\frac{4h^2}{p}\cdot 10^4$		Object	a (AE)	e	T (days)	$\frac{4h^2}{p}\cdot 10^4$
Mercury	0.387099	0.205630	87.9690	2.9591		Vesta	2.768134	0.075705	1682.21	2.9591
Venus	0.723332	0.006773	224.701	2.9591		Jupiter	5.203360	0.048393	4332.59	2.9629
Earth	1.000000	0.016710	365.256	2.9591		Saturn	9.537070	0.054151	10759.2	2.9583
Mars	1.523662	0.093412	686.980	2.9590		Uranus	19.19126	0.047168	30685.4	2.9635
Ceres	2.361348	0.089067	1325.37	2.9591		Neptune	30.06896	0.008586	60189.0	2.9627

The semimajor axis a is given in astronomical units (AU). The astronomical unit is the average distance from the earth to the sun. The period is given in days.

For the smaller planets, the dwarf planet Ceres and the asteroid Vesta, the constant $\frac{4h^2}{p}$ is almost the same.

Small deviations occur in the larger planets. This is because the mass of these planets is so large that it can no longer be neglected in relation to the (much larger) mass of the sun.

If the mass of the object is negligible, then according to Newton's gravitational theory: $\frac{4h^2}{p} = GM$,

whereby:

- G is the gravitational constant, also called Cavendish's constant;
- M is the mass of the sun.

Thus, according to formula (24), Kepler's third law can be formulated as follows:

$$\frac{1}{\pi^2}\frac{h^2}{p} = \frac{a^3}{T^2} = \text{constant} \quad \rightarrow \quad \frac{GM}{4\pi^2} = \frac{a^3}{T^2} = \text{constant} \tag{51}$$

Inserting the constant GM into formula (50) immediately leads to Newton's law of gravitation:

$$F = G\frac{M\,m}{r^2} \tag{52}$$

Determination of the orbit of an asteroid or comet in the orbital plane from 2 given positions

In our solar system, for objects that move in elliptical orbits around the sun, the Gaussian gravitational constant is used, a constant with symbol k, defined as:

$$k = \sqrt{GM} = \frac{2\pi}{\sqrt{1+\mu}} \cdot \frac{a^{3/2}}{T} = 0.01720209895 \qquad (53)$$

The symbol G is here Cavendish's constant, M the solar mass and μ the mass of the object, expressed in solar masses. The semimajor axis a of the elliptical orbit is given in astronomical units (AU) and T is the object's orbital period around the sun in days.
See appendix E for a more detailed explanation of formula (53), in which the exact magnitude of the constant k is also calculated.

For asteroids and comets, the relative mass μ is totally negligible compared to the factor 1: compare formula (51).

According to Kepler's third law in (23) and (24) then holds:

$$k = 2\pi \frac{a^{3/2}}{T} = \frac{2h}{\sqrt{p}} = \text{constant} \quad \text{and} \quad \frac{2\pi}{T} = \frac{2h}{\sqrt{p}} \cdot \frac{1}{a^{3/2}} = \frac{k}{a^{3/2}} \qquad (54)$$

If the asteroid or comet position is known at 2 different times, the orbital parameters can be accurately calculated. These are the semimajor axis a and the eccentricity e of the elliptical orbit as well as the time of perihelion passage t_0.
See figure 6, where 2 positions $P_1(r_1, \theta_1)$ for time t_1 resp. $P_2(r_2, \theta_2)$ for time t_2 are displayed.

If the radius vectors r_1 and r_2 and the true anomalies θ_1 and θ_2 are given, the parameter p and the eccentricity e can be calculated from the orbital equation (6):

$$r_1 = \frac{p}{1 + e\cos\theta_1} \quad \text{and} \quad r_2 = \frac{p}{1 + e\cos\theta_2}$$

$$e = \frac{p - r_1}{r_1\cos\theta_1} = \frac{p - r_2}{r_2\cos\theta_2} \;\rightarrow\; p = \frac{r_1 r_2(\cos\theta_2 - \cos\theta_1)}{r_2\cos\theta_2 - r_1\cos\theta_1} \qquad p = r_1 + r_1 e\cos\theta_1 = r_2 + r_2 e\cos\theta_2 \;\rightarrow\; e = \frac{r_1 - r_2}{r_2\cos\theta_2 - r_1\cos\theta_1} \qquad (55)$$

According to (10) and (54) the semimajor axis a and the orbital time T can be calculated via: $a = \frac{p}{1-e^2}$ and $T = 2\pi\frac{a^{3/2}}{k}$

The time of perihelion passage t_0 requires calculation of the eccentric anomaly according to (30) from one of the true anomalies:

$$\tan\frac{1}{2}E_1 = \sqrt{\frac{1-e}{1+e}}\tan\frac{1}{2}\theta_1 \;\rightarrow\; E_1 = 2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{1}{2}\theta_1\right)$$

The value of t_0 is then according to (32):

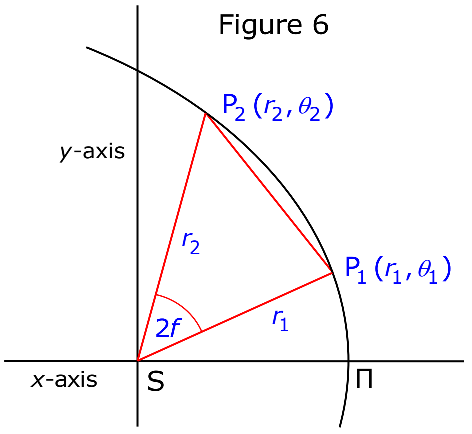
$$t_0 = t_1 - \frac{T}{2\pi}(E_1 - e\sin E_1) \quad \text{or} \quad t_0 = t_1 - \frac{a^{3/2}}{k}(E_1 - e\sin E_1) \qquad (56)$$

When only the second time is given, the data θ_2, E_2 and t_2 should be used in the previous formulas.
This shows that if both radius vectors and true anomalies are given, only one of the times t_1 or t_2 needs to be known.

In astronomical practice the case arises that both times are known, but not both true anomalies are. Carl Friedrich Gauss (1777-1855) showed that only the difference $2f = \theta_2 - \theta_1$ is needed and developed a method for this which is still in use today. It plays an important role in determining the orbits of solar system objects from only 3 given positions, also determining the spatial orientation of the orbital plane. The latter is done by means of 3 additional parameters, which together with the already mentioned parameters form the 6 so-called orbital elements.

In 1802, he was able to calculate the orbit of Ceres, predicted its position and enabled the (re)discovery of this first found asteroid. Thousands of asteroids are now known. This achievement immediately established his reputation as an astronomer, although in 1801 he had already gained fame as a mathematician with his Disquisitiones arithmeticae. In 1807 he was appointed director of the astronomical observatory in Göttingen and professor of astronomy there, a position he held until the end of his life.

Figure 6



In the method mentioned, the area ratio of sector and triangle plays a key role.

This ratio is defined as follows: $$\eta = \frac{A_{\text{sector}}}{A_{\text{triangle}}} = \frac{\text{area ellipse sector } SP_1P_2}{\text{area triangle } SP_1P_2}$$

Suppose $\tau = k\,(t_2 - t_1)$, then holds for the area of the ellipse sector according to (36) and (54):

$$A_{\text{sector}} = A(SP_1P_2) = A(\Pi SP_2) - A(\Pi SP_1) = h(t_2 - t_0) - h(t_1 - t_0) = h(t_2 - t_1) = \frac{1}{2}k\sqrt{p}\,(t_2 - t_1) = \frac{1}{2}\tau\sqrt{p}$$

By taking $\tau = kt$ as a time unit instead of t, the constant k becomes, as it were, incorporated in time. As a result, this constant k no longer appears in the equations.

Consider SP_1 as the base of the triangle SP_1P_2 with length r_1 and draw a perpendicular from P_2 to SP_1. This perpendicular then forms the height of this triangle with length $r_2 \sin 2f$, such that:

$$A_{\text{triangle}} = \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} \cdot r_1 \cdot r_2 \sin 2f$$

So for the area ratio of sector and triangle holds: $$\eta = \frac{A_{\text{sector}}}{A_{\text{triangle}}} = \frac{\tau\sqrt{p}}{r_1 r_2 \sin 2f} \qquad (57)$$

Gauss was faced with the task of calculating the area ratio η from the data r_1, r_2, f and τ. How he managed to solve this problem will be explained on the following pages.

The area ratio sector/triangle

In the formula (57), r_1, r_2, f and τ are known variables, except for the ellipse parameter p. According to (2) applies to this: $p = a(1-e^2)$. If the area ratio η can be calculated, then the magnitude of p follows directly from it.

The eccentricity e can then be calculated via the orbital equations $r = \frac{p}{1+e\cos\theta}$ for both positions, from which the semimajor axis a immediately follows via (2) and the orbital period T via (54). Both eccentric anomalies E_1 and E_2 can then be determined using formula (30), after which the time of perihelion passage t_0 can also be calculated according to (56).

Formulas (29) and (32) are available to calculate the magnitude of η, while according to (54) it holds that $\frac{2\pi}{T} = \frac{k}{a^{3/2}}$:

$$\sqrt{r_1}\sin\frac{1}{2}\theta_1 = \sqrt{a(1+e)}\sin\frac{1}{2}E_1 \qquad \sqrt{r_1}\cos\frac{1}{2}\theta_1 = \sqrt{a(1-e)}\cos\frac{1}{2}E_1 \qquad E_1 - e\sin E_1 = \frac{k}{a^{3/2}}(t_1 - t_0)$$

$$\sqrt{r_2}\sin\frac{1}{2}\theta_2 = \sqrt{a(1+e)}\sin\frac{1}{2}E_2 \qquad \sqrt{r_2}\cos\frac{1}{2}\theta_2 = \sqrt{a(1-e)}\cos\frac{1}{2}E_2 \qquad E_2 - e\sin E_2 = \frac{k}{a^{3/2}}(t_2 - t_0)$$

Combining the above formulas yields the following four equations (see appendix F for the derivation):
The following variables were introduced: $\theta_2 - \theta_1 = 2f$ $\quad E_2 - E_1 = 2g$ $\quad E_2 + E_1 = 2G$ $\quad \tau = k(t_2 - t_1)$

- $\sqrt{r_1 r_2}\cos f = a(\cos g - e\cos G)$ (58)
- $\sqrt{r_1 r_2}\sin f = \sqrt{ap}\sin g$ (59)
- $r_1 + r_2 = 2a - 2ae\cos g\cos G$ (60)
- $\frac{\tau}{a^{3/2}} = 2g - 2e\sin g\cos G$ (61)

The factor $e\cos G$ in (60) and (61) can be eliminated by substitution of $e\cos G = \cos g - \frac{\sqrt{r_1 r_2}\cos f}{a}$ according to equation (58):

$$r_1 + r_2 - 2a = 2a\cos g(\cos g - \frac{\sqrt{r_1 r_2}\cos f}{a}) \quad \rightarrow \quad r_1 + r_2 - 2\sqrt{r_1 r_2}\cos f\cos g = 2a - 2a\cos^2 g = 2a\sin^2 g \qquad (62)$$

$$\frac{\tau}{a^{3/2}} = 2g - 2\sin g(\cos g - \frac{\sqrt{r_1 r_2}\cos f}{a}) = 2g - \sin 2g + \frac{2\sqrt{r_1 r_2}\cos f\sin g}{a} \qquad (63)$$

The factor $\sqrt{p}$ in (57) can be eliminated by substitution of $\sqrt{p} = \frac{\sqrt{r_1 r_2}\sin f}{\sqrt{a}\sin g}$ according to equation (59):

$$\eta = \frac{\tau\sqrt{p}}{r_1 r_2 \sin 2f} = \frac{\tau\sqrt{p}}{2r_1 r_2 \sin f \cos f} = \frac{\tau \cdot \frac{\sqrt{r_1 r_2}\sin f}{\sqrt{a}\sin g}}{2r_1 r_2 \sin f \cos f} = \frac{\tau}{2\sqrt{a}\sqrt{r_1 r_2}\cos f \sin g} \qquad (64)$$

The unknowns in equations (62), (63) and (64) are a, g and η. Eliminating a leaves 2 equations in g and η.

From the equation (64) it follows that: $a = \frac{\tau^2}{4\eta^2 r_1 r_2 \cos^2 f \sin^2 g}$ and $a^{3/2} = \frac{\tau^3}{8\eta^3(\sqrt{r_1 r_2})^3 \cos^3 f \sin^3 g}$

Substitution of a in equation (62) gives: $r_1 + r_2 - 2\sqrt{r_1 r_2}\cos f \cos g = \frac{\tau^2}{2\eta^2 r_1 r_2 \cos^2 f}$ $\rightarrow$

$$\rightarrow\ 4\sqrt{r_1 r_2}\cos f\left(\frac{r_1 + r_2}{4\sqrt{r_1 r_2}\cos f} - \frac{1}{2}\cos g\right) = \frac{\tau^2}{2\eta^2 r_1 r_2 \cos^2 f} \quad \rightarrow \quad \frac{r_1 + r_2}{4\sqrt{r_1 r_2}\cos f} - \frac{1}{2}\cos g = \frac{\tau^2}{8\eta^2(\sqrt{r_1 r_2})^3 \cos^3 f}$$

In the last equation $\cos g$ is replaced by $1 - 2\sin^2 \frac{1}{2} g$:

$$\frac{r_1 + r_2}{4\sqrt{r_1 r_2}\cos f} - \frac{1}{2} + \sin^2 \frac{1}{2} g = \frac{\tau^2}{8\eta^2(\sqrt{r_1 r_2})^3 \cos^3 f} \qquad (65)$$

Subsitution of $a^{3/2}$ resp. a in equation (63) gives:

$$\frac{8\eta^3(\sqrt{r_1 r_2})^3 \cos^3 f \sin^3 g}{\tau^2} = 2g - \sin 2g + \frac{8\eta^2(\sqrt{r_1 r_2})^3 \cos^3 f \sin^3 g}{\tau^2} \qquad (66)$$

Equations (65) and (66) can be greatly simplified by introducing the following variables μ and λ:

$$\mu = \frac{\tau^2}{(2\sqrt{r_1 r_2}\cos f)^3} \quad \text{and} \quad \lambda = \frac{r_1 + r_2}{4\sqrt{r_1 r_2}\cos f} - \frac{1}{2} \qquad (67)$$

Subsitution of μ and λ in (65): $\lambda + \sin^2 \frac{1}{2} g = \frac{\mu}{\eta^2}$ $\rightarrow$ $\eta^2 = \frac{\mu}{\lambda + \sin^2 \frac{1}{2} g}$ (68)

Substitution of μ in (66): $\frac{\eta^3}{\mu}\sin^3 g = 2g - \sin 2g + \frac{\eta^2}{\mu}\sin^3 g$ $\rightarrow$ $\eta^3 - \eta^2 = \frac{\mu(2g - \sin 2g)}{\sin^3 g}$ (69)

Solving η from the 2 equations

The area ratio of sector and triangle η from the equations $\eta^2 = \dfrac{\mu}{\lambda + \sin^2 \frac{1}{2} g}$ and $\eta^3 - \eta^2 = \dfrac{\mu(2g - \sin 2g)}{\sin^3 g}$ can be solved by eliminating g, which is simplified by dividing the second equation (69) by the first (68).

Suppose that $\xi = \sin^2 \frac{1}{2} g$ and $X(\xi) = \dfrac{2g - \sin 2g}{\sin^3 g}$, then the following equation arises: $\eta = 1 + X(\xi)(\lambda + \xi)$ (70)

With a starting value of g, ξ and $X(\xi)$ can then be calculated, whereby a first approximation of η is obtained via (70).

An improved value of ξ is then calculated via (68) with $\eta^2 = \dfrac{\mu}{\lambda + \xi}$, from which ξ follows with: $\xi = \dfrac{\mu}{\eta^2} - \lambda$ (71)

Equations (70) and (71) are then used alternately until η is known with sufficient accuracy.

If a weakly eccentric orbit is assumed, $g = f$ can be taken as the starting value for the iteration. After all, $g = \frac{1}{2}(E_2 - E_1) = \frac{1}{2}(\theta_2 - \theta_1) = f$ holds for a circular orbit, because $e = 0$ and $\theta = E$. At high eccentricity, the orbit looks more like a parabola (as with comets), so $g = 0$ can serve as a starting value for the iteration. See also appendix K for the parabola as limiting case of the ellipse.

As soon as an iteration step via (71) has yielded a new value of ξ, new values of g, $\sin g$ and $\sin 2g$ must be recalculated. This is to obtain an improved value of $X(\xi)$ at (70), whereby the relations (73) can be used:

$$\cos g = 1 - 2\xi \qquad \sin g = 2\sqrt{\xi(1-\xi)} \qquad \sin 2g = 2 \sin g \cos g \qquad g = \arcsin(\sin g)$$

These calculation steps can be avoided by expressing $X(\xi) = \dfrac{2g - \sin 2g}{\sin^3 g}$ in a power series as a function of $\xi = \sin^2 \frac{1}{2} g$, where $X(\xi)$ is represented by the series $X(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + a_5\xi^5 + \ \ldots$ (72)

When publishing his book Theoria Motus Corporum Coelestium in 1809, Gauss was the first to mention this polynomial.

The coefficients a_n of this series will be derived below.

To this end, the derivatives from X and ξ with respect to g are first to be determined:

$$\frac{dX}{dg}\cdot\sin^3 g + 3X\sin^2 g\cos g = 2 - 2\cos g = 4\sin^2 g \quad\rightarrow\quad \frac{dX}{dg}\cdot\sin g + 3X\cos g = 4 \quad\text{and}\quad \frac{d\xi}{dg} = \sin\frac{1}{2}g\cos\frac{1}{2}g = \frac{1}{2}\sin g$$

Substituting of $\frac{dX}{dg} = \frac{dX}{d\xi}\cdot\frac{d\xi}{dg}$ into the above equation then gives: $\frac{1}{2}\frac{dX}{d\xi}\cdot\sin^2 g + 3X\cos g = 4$

Furthermore: $\cos g = 1 - 2\sin^2\frac{g}{2} = 1 - 2\xi$ and $\sin^2 g = 1 - \cos^2 g = 1 - (1-2\xi)^2 = 4\xi(1-\xi)$ (73)

Substituting $\cos g$ and $\sin^2 g$ yields the final relationship between X and ξ: $2\xi(1-\xi)\frac{dX}{d\xi} + 3X(1-2\xi) - 4 = 0$ (74)

The series (72): $X = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + a_5\xi^5 + a_6\xi^5 + \ \ldots$ has as derivative: $\frac{dX}{d\xi} = a_1 + 2a_2\xi + 3a_3\xi^2 + 4a_4\xi^3 + 5a_5\xi^4 + \ \ldots$

Inserting both polynomials into the differential equation (74) and arranging the terms with the same exponent gives:

$$+2\xi\frac{dX}{d\xi} = +2a_1\xi + 4a_2\xi^2 + 6a_3\xi^3 + 8a_4\xi^4 + 10a_5\xi^5 + \ \ldots$$

$$-2\xi^2\frac{dX}{d\xi} = -2a_1\xi^2 - 4a_2\xi^3 - 6a_3\xi^4 - 8a_4\xi^5 + \ \ldots$$

$$+3X = +3a_0 + 3a_1\xi + 3a_2\xi^2 + 3a_3\xi^3 + 3a_4\xi^4 + 3a_5\xi^5 + \ \ldots$$

$$-6X\xi - 4 = -4 \quad -6a_0\xi - 6a_1\xi^2 - 6a_2\xi^3 - 6a_3\xi^4 - 6a_4\xi^5 + \ \ldots = 0$$

By comparing the coefficients at corresponding powers of ξ the following can be concluded:

$$3a_0 - 4 = 0 \quad\rightarrow\quad a_0 = \frac{4}{3}$$

$$-6a_0 + 5a_1 = 0 \quad\rightarrow\quad a_1 = a_0\cdot\frac{6}{5}$$

$$-8a_1 + 7a_2 = 0 \quad\rightarrow\quad a_2 = a_0\cdot\frac{6}{5}\cdot\frac{8}{7} = a_0\cdot\frac{48}{35}$$

$$-10a_2 + 9a_3 = 0 \quad\rightarrow\quad a_3 = a_0\cdot\frac{6}{5}\cdot\frac{8}{7}\cdot\frac{10}{9} = a_0\cdot\frac{32}{21} \qquad \text{Generally:}\quad a_n = a_0\cdot\prod_{i=1}^{n}\frac{2i+4}{2i+3}$$

$$-12a_3 + 11a_4 = 0 \quad\rightarrow\quad a_4 = a_0\cdot\frac{6}{5}\cdot\frac{8}{7}\cdot\frac{10}{9}\cdot\frac{12}{11} = a_0\cdot\frac{128}{77}$$

$$-14a_4 + 13a_5 = 0 \quad \rightarrow \quad a_5 = a_0 \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{10}{9} \cdot \frac{12}{11} \cdot \frac{14}{13} = a_0 \cdot \frac{256}{143}$$

...

Result: $$X(\xi) = \frac{4}{3}(1 + \frac{6}{5}\xi + \frac{48}{35}\xi^2 + \frac{32}{21}\xi^3 + \frac{128}{77}\xi^4 + \frac{256}{143}\xi^5 + \ldots) \quad (75)$$

The coefficients a_n are greater than 1 and slowly increase in size. This series will converge only at very small values of ξ.

Because $\xi = \sin^2 \frac{1}{2} g = \sin^2 \frac{1}{4}(E_2 - E_1)$, this power series is only suitable for 2 positions that are very close to each other.

Gauss realized that the series (75) resembles the geometric series: $\frac{4}{3}(1 + \frac{6}{5}\xi + (\frac{6}{5}\xi)^2 + (\frac{6}{5}\xi)^3 + \ \ldots) = \frac{4}{3} \cdot \frac{1}{1 - 6/5 \cdot \xi} < X(\xi)$

and corrected ξ with a small factor ν, so that $X(\xi) = \frac{4}{3} \cdot \frac{1}{1 - 6/5(\xi - \nu)}$, where ν can be developed in a polynomial as a function of ξ.

Because $\nu = \frac{10}{9X(\xi)} - \frac{5}{6} + \xi$, the sequence development of $\frac{1}{X(\xi)}$ is required. The derivation of the polynomial coefficients of this last series is more complicated than that of $X(\xi)$, but the result is as follows: $$\frac{1}{X(\xi)} = \frac{3}{4} - \frac{9}{10}\xi + \frac{9}{175}\xi^2 + \frac{26}{875}\xi^3 + \frac{6228}{336875}\xi^4 + \ldots \quad (76)$$

Plugging in the last series in the equation for ν then gives: $$\nu(\xi) = \frac{2}{35}\xi^2 + \frac{52}{1575}\xi^3 + \frac{1384}{67375}\xi^4 + \frac{59088}{4379375}\xi^5 + \ldots \quad (77)$$

This shows that $\nu(\xi)$ contains only terms of the second and higher order. The value of $X(\xi)$ is obtained by filling in ν in the formula $X(\xi) = \frac{4}{3} \cdot \frac{1}{1 - 6/5(\xi - \nu)}$ after calculation with series (77). See appendix G for more polynomial coefficients and their derivation.

It is more convenient to take the series (76) instead of the series (77), so that $X(\xi)$ can simply be obtained as a reciprocal value.

Cutting off both series at the n^{th} term produces an error $|\Delta X(\xi)|$, that depends only on the size of $\xi = \sin^2 \frac{1}{2} g$.

The minimum required number of terms of both last series as a function of g at an accuracy of $|\Delta X(\xi)| < 5 \cdot 10^{-9}$ is as follows:

g (degrees)		5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85
Number of terms n		3	4	4	5	6	6	7	7	8	9	10	11	12	13	15	17	19

An alternate power series for X

According to equation (74) holds: $2\xi(1-\xi)\dfrac{dX}{d\xi}+3X(1-2\xi)-4=0$, where $\xi=\sin^2\frac{1}{2}g$ with $\frac{1}{2}g<\frac{1}{4}\pi$, so $0<\xi<\frac{1}{2}$

A slightly more convergent power series can be obtained by taking $\tan^2\frac{1}{2}g$ as a parameter, instead of $\sin^2\frac{1}{2}g$.

Set: $$\zeta=\tan^2\frac{1}{2}g=\frac{\sin^2\frac{1}{2}g}{\cos^2\frac{1}{2}g}=\frac{\sin^2\frac{1}{2}g}{1-\sin^2\frac{1}{2}g}=\frac{\xi}{1-\xi} \quad \text{with} \quad 0<\zeta<1 \qquad (78)$$

As is easy to deduce, according to definition (78) and the differential equation (74), the following relations between ξ and ζ exists:

$$\xi=\frac{\zeta}{1+\zeta} \qquad 1-\xi=\frac{1}{1+\zeta} \qquad \frac{d\zeta}{d\xi}=\frac{(1-\xi)+\xi}{(1-\xi)^2}=\frac{1}{(1-\xi)^2}=(1+\zeta)^2 \qquad \frac{dX}{d\xi}=\frac{d\zeta}{d\xi}\cdot\frac{dX}{d\zeta}=(1+\zeta)^2\frac{dX}{d\zeta} \qquad (79)$$

By substitution of these relations, equation (74) can be converted to:

$$2\frac{\zeta}{(1+\zeta)^2}\cdot(1+\zeta)^2\frac{dX}{d\zeta}+3X(1-\frac{2\zeta}{1+\zeta})-4=0 \qquad \rightarrow \qquad 2\zeta(1+\zeta)\frac{dX}{d\zeta}+3X(1-\zeta)-4\zeta-4=0 \qquad (80)$$

The variable X can be developed in a polynomial as a function of variable ζ according to the method applied on page 19.

Represent this power series as follows: $X(\zeta)=b_0+b_1\zeta+b_2\zeta^2+b_3\zeta^3+b_4\zeta^4+b_4\zeta^4+b_5\zeta^5+\ ...$ (81)

The derivative of this series is then: $\dfrac{dX}{d\zeta}=b_1+2b_2\zeta+3b_3\zeta^2+4b_4\zeta^3+5b_5\zeta^4+\ ...$

The terms b_n are derived by substitution of the series and its derivative in the differential equation (80):

$$+2\zeta\frac{dX}{d\zeta}=+2b_1\zeta+4b_2\zeta^2+6b_3\zeta^3+8b_4\zeta^4+10b_5\zeta^5+\ ...$$

$$+2\zeta^2\frac{dX}{d\zeta}=+2b_1\zeta^2+4b_2\zeta^3+6b_3\zeta^4+8b_4\zeta^5+\ ...$$

$$+3X=+3b_0+3b_1\zeta+3b_2\zeta^2+3b_3\zeta^3+3b_4\zeta^4+3b_5\zeta^5+\ ...$$

$$-3X\zeta=-3b_0\zeta-3b_1\zeta^2-3b_2\zeta^3-3b_3\zeta^4-3b_4\zeta^5+\ ...$$

$$-4\zeta-4=-4-4\zeta=0$$

By comparing the coefficients at corresponding powers of ζ the following can be concluded:

$$3b_0 - 4 = 0 \quad\rightarrow\quad b_0 = +4\cdot\frac{1}{3} = +8\cdot\frac{1}{6} = +\frac{8}{6}$$

$$5b_1 - 3b_0 = 4 \quad\rightarrow\quad b_1 = +\frac{6}{5}b_0 = -8\cdot\frac{3}{-1\cdot1\cdot3\cdot5} = +\frac{8}{5}$$

$$7b_2 - 1b_1 = 0 \quad\rightarrow\quad b_2 = +\frac{1}{7}b_1 = +8\cdot\frac{3}{1\cdot3\cdot5\cdot7} = +\frac{8}{35}$$

$$9b_3 + 1b_2 = 0 \quad\rightarrow\quad b_3 = -\frac{1}{9}b_2 = -8\cdot\frac{3}{3\cdot5\cdot7\cdot9} = -\frac{8}{315}$$

$$11b_4 + 3b_3 = 0 \quad\rightarrow\quad b_4 = -\frac{3}{11}b_3 = +8\cdot\frac{3}{5\cdot7\cdot9\cdot11} = +\frac{8}{1155}$$

$$13b_5 + 5b_4 = 0 \quad\rightarrow\quad b_5 = -\frac{5}{13}b_4 = -8\cdot\frac{3}{7\cdot9\cdot11\cdot13} = -\frac{8}{3003}$$

...

for $n \geq 1$: $\quad b_n = 8\cdot\frac{3\cdot(-1)^n}{(2n-3)(2n-1)(2n+1)(2n+3)}$

for $n \geq 2$: $\quad b_{n+1} = -b_n\cdot\frac{2n-3}{2n+5}$

Result: $$X(\zeta) = 8\cdot\left(\frac{1}{6} + \frac{1}{5}\zeta + \frac{1}{35}\zeta^2 - \frac{1}{315}\zeta^3 + \frac{1}{1155}\zeta^4 - \frac{1}{3003}\zeta^5 - \ldots\right) \quad \text{with} \quad \zeta = \frac{\xi}{1-\xi} = \tan^2\frac{1}{2}g \qquad (82)$$

Since the determination of η only makes sense at $2g < \pi$ (180 degrees), $\zeta = \tan^2\frac{1}{2}g < 1$, so the series $X(\zeta)$ always converges.

When the series $X(\zeta)$ is truncated at the n^{th} term b_n, the maximum error $|\Delta X|$ in the series can be estimated by giving all terms following b_n a coefficient equal to b_n. The truncation error is then somewhat larger than the actual value.

If b_n and all subsequent coefficients would be neglected, the truncation error is less than $b_n \cdot \zeta^n$ (alternating series).

The truncated part can be represented in this way as: $\left|b_n\zeta^n - b_n\zeta^{n+1} + b_n\zeta^{n+2} - b_n\zeta^{n+3} + \ldots\right| = |b_n|\cdot\zeta^n\cdot\sum_{i=0}^{\infty}(-1)^i\cdot\zeta^i$

with $|b_n| = 8\cdot\frac{3}{(2n-3)(2n-1)(2n+1)(2n+3)}$ and the geometric series $\sum_{i=0}^{\infty}(-1)^i\cdot\zeta^i = \frac{1}{1+\zeta}$, with $\zeta < 1$.

Thus the following applies to the truncation error: $$|\Delta X| < \frac{24}{(2n-3)(2n-1)(2n+1)(2n+3)}\cdot\frac{\zeta^n}{1+\zeta} \qquad (83)$$

The number of terms to be taken in the calculation of $X(\zeta)$ depends first of all on the required accuracy, but on the magnitude of $\zeta = \tan^2 \frac{1}{2} g$ as well. The following table shows how many terms n of the series are at least required as a function of g in degrees at an accuracy of $|\Delta X| < 5 \cdot 10^{-9}$, calculated with formula (83):

g (degrees)		5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85
Number of terms n		3	4	4	5	5	5	6	7	7	8	9	10	11	13	16	21	31

The convergence of the iteration

The convergence of the iteration in the Gauss method, as described on page 18, leaves something to be desired at significant values of g (and of f), especially at larger eccentricities of the ellipse orbit.

The cause can be found in the functions (70) and (71), here indicated as η_a and η_b: $\eta_a(\xi) = 1 + X(\xi)(\lambda + \xi)$ and $\eta_b(\xi) = \sqrt{\frac{\mu}{\lambda + \xi}}$

For $\eta_a(\xi)$ holds: $\lim_{\xi \to 0} \eta_a(\xi) = \lim_{\xi \to 0} 1 + X(\xi)(\lambda + \xi) = 1 + \frac{4}{3}\lambda$. See appendix H for proof that $X(0) = \frac{4}{3}$; this also follows from (74) and (75).

The first function is monotonically increasing with minimum value $\eta_a(0) = 1 + \frac{4}{3}\lambda$ and maximum of $\eta_a(\frac{1}{2}) = \lim_{\xi \to \frac{1}{2}} 1 + X(\xi)(\lambda + \xi) = \infty$.

The second function is monotonically descending with minimum $\eta_b(\frac{1}{2}) = \sqrt{\frac{2\mu}{2\lambda + 1}}$ and maximum of $\eta_b(0) = \sqrt{\frac{\mu}{\lambda}}$.

Because $\eta_b(0) > \eta_a(0)$ the two functions always have an intersection point in the interval $0 < \xi < \frac{1}{2}$. See appendix I for a proof.

These intersections, where $\eta_a(\xi) = \eta_b(\xi)$, represent the solutions η for all ξ in mentioned interval (indeed $g < 90^o$ and $f < 90^o$).

The iterative method of alternately applying equations (70) and (71) fails if, during the iteration, ever larger values of ξ arise, which in turn lead to larger values η (divergence). It can even lead to the fact that a value of η, calculated according to (70) with $\eta = 1 + X(\xi)(\lambda + \xi)$, can be greater than $\eta_b(0) = \sqrt{\frac{\mu}{\lambda}}$. In such a case $\eta^2 > \frac{\mu}{\lambda}$, so that $\lambda > \frac{\mu}{\eta^2}$ and $\xi = \frac{\mu}{\eta^2} - \lambda < 0$. See appendix J for further explanation with some examples.

This drawback is overcome with Newton's method for a system of 2 simultaneous equations with two unknowns.

Determining the area ratio using Newton's method

To solve the system of two equations (70) and (71) with Newton's method, the following functions are defined:

$$f(\xi,\eta) = \eta(\xi) - 1 - X(\xi)(\lambda + \xi) = 0 \qquad \text{and} \qquad g(\xi,\eta) = \eta^2(\xi) - \frac{\mu}{\lambda + \xi} = 0 \tag{84}$$

The solution (ξ_0, η_0) can be obtained iteratively so that approximately $f(\xi_0,\eta_0) = g(\xi_0,\eta_0) = 0$ is satisfied.

To this end, both functions are being developed in a Taylor series in the vicinity of the solution (ξ_0,η_0), with $\xi_0 = \xi + \Delta\xi$ and $\eta_0 = \eta + \Delta\eta$. Here (ξ,η) is the estimated solution and $(\Delta\xi,\Delta\eta)$ the correction to it. Only first-order partial derivatives are used, while the higher-order terms are being ignored:

$$f(\xi_0,\eta_0) = f(\xi,\eta) + \frac{\partial f}{\partial \xi}\cdot\Delta\xi + \frac{\partial f}{\partial \eta}\cdot\Delta\eta = 0 \qquad \text{and} \qquad g(\xi_0,\eta_0) = g(\xi,\eta) + \frac{\partial g}{\partial \xi}\cdot\Delta\xi + \frac{\partial g}{\partial \eta}\cdot\Delta\eta = 0 \tag{85}$$

The following applies to the partial derivatives:

$$\frac{\partial f}{\partial \xi} = -\frac{\mathrm{d}X(\xi)}{\mathrm{d}\xi}\cdot(\xi+\lambda) - X(\xi) \qquad \frac{\partial f}{\partial \eta} = 1 \qquad \frac{\partial g}{\partial \xi} = +\frac{\mu}{(\xi+\lambda)^2} \qquad \frac{\partial g}{\partial \eta} = 2\eta \tag{86}$$

For example, using the series (82), $\frac{\mathrm{d}X(\xi)}{\mathrm{d}\xi}$ can be calculated according to formula (79):

$$\frac{\mathrm{d}X}{\mathrm{d}\xi} = (1+\zeta)^2\frac{\mathrm{d}X}{\mathrm{d}\zeta} = (1+\zeta)^2(b_1 + 2b_2\zeta + 3b_3\zeta^2 + 4b_4\zeta^3 + 5b_5\zeta^4 + \ \ldots) \qquad \text{with} \qquad \zeta = \frac{\xi}{1-\xi} \qquad \text{and} \qquad b_n \ \text{ of the series (82).}$$

The corrections $\Delta\xi$ and $\Delta\eta$ to the estimated values ξ_0 and η_0 are solved from the system of 2 linear equations (85).
From (85) and (86) it follows the Jacobi-matrix of partial derivatives J, the matrix of function values B and the solution matrix U:

$$J = \begin{bmatrix} \frac{\partial f}{\partial \xi} & \frac{\partial f}{\partial \eta} \\ \frac{\partial g}{\partial \xi} & \frac{\partial g}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -\frac{\partial X(\xi)}{\partial \xi}\cdot(\xi+\lambda) - X(\xi) & 1 \\ \frac{\mu}{(\xi+\lambda)^2} & 2\eta \end{bmatrix} \qquad B = \begin{bmatrix} -f(\xi,\eta) \\ -g(\xi,\eta) \end{bmatrix} \qquad U = \begin{bmatrix} \Delta\xi \\ \Delta\eta \end{bmatrix} \tag{87}$$

The solution for $(\Delta\xi,\Delta\eta)$ then follows via $U = J^{-1}B$, where J^{-1} is the inverse Jacobi matrix. The corrected values $\xi + \Delta\xi$ and $\eta + \Delta\eta$ serve as a starting point for improved values J and B, which can be reused. In this way the solution (ξ_0,η_0) can be found in a very limited number of iterations, with $\Delta\xi$ and $\Delta\eta$ approaching zero, depending on the required accuracy.

Determination of the orbital parameters

Once the sector/triangle area ratio η is known, the ellipse parameter p, the eccentricity e, the semimajor axis a, and time of perihelion passage t_0 can be calculated.

First of all, the parameter p is determined by applying formula (57):

$$p = \left(\frac{r_1 r_2 \sin 2f}{\tau}\right)^2 \cdot \eta^2$$

The calculation of the eccentricity is based on the use of the orbital equations (6) for both positions, requiring only the sine and cosine of the angle $2f = \theta_2 - \theta_1$. To simplify the formulas, auxiliary parameters q_1 and q_2 are introduced:

$$r_1 = \frac{p}{1 + e\cos\theta_1} \quad \text{with} \quad q_1 = e\cos\theta_1 = \frac{p}{r_1} - 1 \qquad \text{and} \qquad r_2 = \frac{p}{1 + e\cos\theta_2} \quad \text{with} \quad q_2 = e\cos\theta_2 = \frac{p}{r_2} - 1 \tag{88}$$

Then applies: $\quad e\cos\theta_2 = q_2 = e\cos(\theta_1 + 2f) = e\cos\theta_1 \cos 2f - e\sin\theta_1 \sin 2f$

Substitution of $e\cos\theta_1 = q_1$ in the previous formula gives: $\quad e\sin\theta_1 = \dfrac{q_1 \cos 2f - q_2}{\sin 2f}$

In an analogous way it is easy to derive that:

$$e\sin\theta_2 = \frac{q_1 - q_2 \cos 2f}{\sin 2f} \tag{89}$$

The eccentricity can now be calculated from:

$$e^2 = (e\sin\theta_1)^2 + (e\cos\theta_1)^2 = \left(\frac{q_1\cos 2f - q_2}{\sin 2f}\right)^2 + q_1^2 = \frac{q_1^2 - 2q_1q_2\cos 2f + q_2^2}{\sin^2 2f} \tag{90}$$

The true anomalies can be calculated in the correct quadrant via the definitions (88) and the two formulas (89) with:

$$\tan\theta_1 = \frac{e\sin\theta_1}{e\cos\theta_1} = \frac{q_1\cos 2f - q_2}{q_1 \sin 2f} \qquad \text{and} \qquad \tan\theta_2 = \frac{e\sin\theta_2}{e\cos\theta_2} = \frac{q_1 - q_2\cos 2f}{q_2\sin 2f} \tag{91}$$

If one of the two true anomalies is known, the other follows from the relation $\theta_2 - \theta_1 = 2f$ as well.

The rectangular coordinates can then be calculated using the formulas (5) with:

$$x_1 = r_1\cos\theta_1 = \frac{r_1 q_1}{e} \qquad y_1 = r_1\sin\theta_1 = \frac{r_1(q_1\cos 2f - q_2)}{e\sin 2f} \qquad x_2 = r_2\cos\theta_2 = \frac{r_2 q_2}{e} \qquad y_2 = r_2\sin\theta_2 = \frac{r_2(q_1 - q_2\cos 2f)}{e\sin 2f} \tag{92}$$

The square of the factor $2\sqrt{r_1 r_2}\cos f$ in the constants μ and λ in the determination of the sector/triangle area ratio in (67) is directly related to the rectangular coordinates in the following way:

$$\kappa^2 = 4r_1r_2\cos^2 f = 4r_1r_2\cos^2\frac{1}{2}(\theta_2 - \theta_1) = 2r_1r_2(1 + \cos(\theta_2 - \theta_1)) = 2(r_1r_2 + r_1\cos\theta_1 \cdot r_2\cos\theta_2 + r_1\sin\theta_1 \cdot r_2\sin\theta_2) = 2(r_1r_2 + x_1x_2 + y_1y_2) \tag{93}$$

Expressed in the variable κ, the formulas for μ and λ then are: $\mu = \frac{\tau^2}{\kappa^3}$ and $\lambda = \frac{r_1 + r_2}{2\kappa} - \frac{1}{2}$ (94)

The semimajor axis a can be calculated according to formula (10): $a = \frac{p}{1 - e^2}$

The orbital period T can be calculated with formula (54): $T = 2\pi \frac{a^{3/2}}{k}$

According to (30) holds for the eccentric anomaly: $\tan\frac{1}{2}E = \sqrt{\frac{1-e}{1+e}} \tan\frac{1}{2}\theta$, where $\tan\frac{1}{2}\theta = \frac{\sin\theta}{1+\cos\theta} = \frac{e\sin\theta}{e + e\cos\theta}$

E_1 resp. E_2 can now be calculated from: $\tan\frac{1}{2}E_1 = \sqrt{\frac{1-e}{1+e}} \cdot \frac{q_1 \cos 2f - q_2}{(e + q_1)\sin 2f}$ and $\tan\frac{1}{2}E_2 = \sqrt{\frac{1-e}{1+e}} \cdot \frac{q_1 - q_2 \cos 2f}{(e + q_2)\sin 2f}$ (95)

If one of the eccentric anomalies is calculated, the other also follows from the relation $E_2 - E_1 = 2g$

The passage time of the celestial body through the perihelion is according to formula (56): $t_0 = t_1 - \frac{a^{3/2}}{k}(E_1 - e\sin E_1)$ (96)

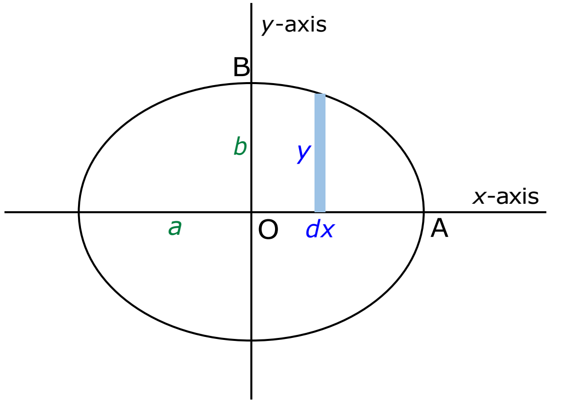
Optionally, the factor $e\sin E_1$ according to the formulas (14), (28), (88) and (89) can be calculated via:

$$e\sin E_1 = e\frac{r_1 \sin\theta_1}{b} = \frac{r_1\sqrt{1-e^2}}{p} \cdot e\sin\theta_1 = \frac{\sqrt{1-e^2}}{1+q_1} \cdot e\sin\theta_1 = \frac{\sqrt{1-e^2}}{1+q_1} \cdot \frac{q_1\cos 2f - q_2}{\sin 2f} \quad (97)$$

Appendix A (see area ellipse)

Calculating the area of an ellipse.

The area of the ellipse sector in the first quadrant is calculated, where the center of the ellipse coincides with the origin O of the coordinate system (see figure opposite).



The equation of the ellipse in rectangular coordinates: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Parametric equation of the ellipse: $x = a\cos t$ and $y = b\sin t$

where $\mathrm{d}x = -a\sin t\,\mathrm{d}t$, while the parameter t ranges from $\frac{1}{2}\pi$ to 0.

The area A_1 of the ellipse sector OAB is then:

$$A_1 = \int_0^a y\,\mathrm{d}x = \int_{\pi/2}^{0} -ab\sin^2 t\,\mathrm{d}t = -\frac{1}{2}ab\int_{\pi/2}^{0}(1-\cos 2t)\,\mathrm{d}t = -\frac{1}{2}ab\left(t - \frac{1}{2}\sin 2t\right)\Big|_{\pi/2}^{0} = \frac{1}{4}\pi ab$$

The total area of the ellipse is four times as large as A_1, so: $A = 4A_1 = \pi ab$

Appendix B (see equation (28))

Calculation of $r\sin\theta$ using formulas (22) and (23), namely: $r\cos\theta = a(\cos E - e)$ and $r = a(1 - e\cos E)$.

$$r^2\sin^2\theta = r^2(1-\cos^2\theta) = r^2 - r^2\cos^2\theta = r^2 - a^2(\cos E - e)^2 = a^2(1-e\cos E)^2 - a^2(\cos E - e)^2 =$$

$$= a^2 - 2a^2e\cos E + a^2e^2\cos^2 E - a^2\cos^2 E + 2a^2e\cos E - a^2e^2 =$$

$$= a^2(1-\cos^2 E) - a^2e^2(1-\cos^2 E) = a^2(1-\cos^2 E)(1-e^2) = a^2(1-e^2)\sin^2 E$$

$$r\sin\theta = a\sqrt{1-e^2}\,\sin E = b\sin E$$

Appendix C (see Kepler's equation)

Kepler's task was to express the area of the ellipse sector ΠSP in the eccentric anomaly E .

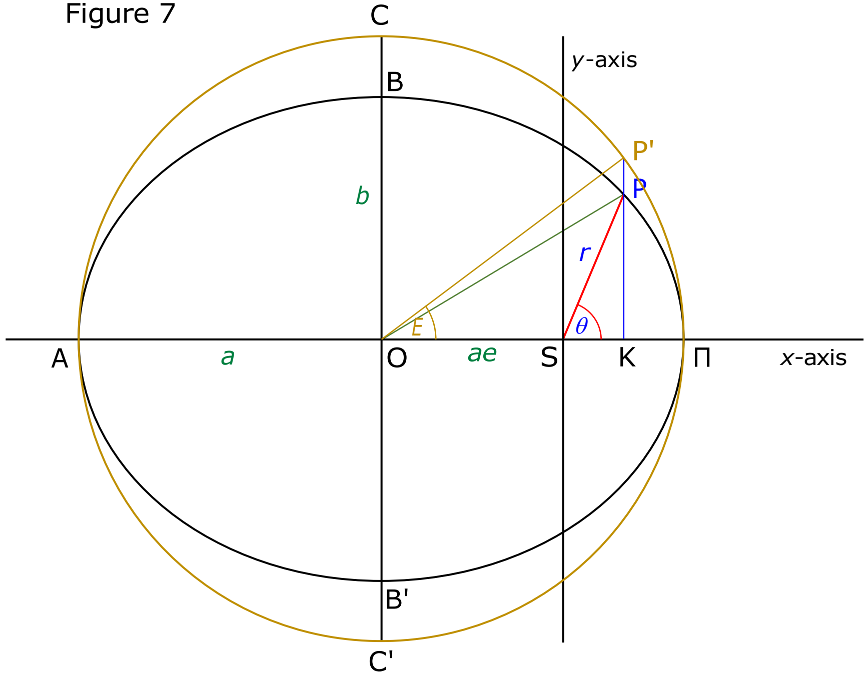
He proceeded as follows (see figure 7):

area circle sector ΠOP' : $A(\Pi OP') = \frac{1}{2}a^2 E$

area ellipse sector ΠOP : $A(\Pi OP) = \frac{b}{a} A(\Pi OP') = \frac{1}{2} ab\, E$

The ratio $\frac{b}{a}$ follows from: $\frac{KP}{KP'} = \frac{OB}{OC} = \frac{b}{a}$ with $KP' = a \sin E$

Figure 7



The following also applies to the area of the ellipse sector ΠOP :

$A(\Pi OP) = A(\Pi SP) + A(OSP)$

For the area of triangle OSP holds:

$$A(OSP) = \frac{1}{2} OS \cdot KP = \frac{1}{2} OS \cdot \frac{b}{a} KP' = \frac{1}{2} ae \cdot \frac{b}{a} a \sin E = \frac{1}{2} eab \sin E$$

The area of the ellipse sector ΠSP can therefore be calculated as follows:

$$A(\Pi SP) = A(\Pi OP) - A(OSP) = \frac{1}{2} ab\, E - \frac{1}{2} eab \sin E = \frac{1}{2} ab (E - e \sin E)$$

Suppose that the time elapsed since perihelion passage, required to reach the position P, is equal to $t - t_0$, while the orbital period of the planet is T .

Then the conclusion with regard to the areas covered by the radius vector is:

elapsed time $t - t_0$: $A(\Pi SP) = \frac{1}{2} ab (E - e \sin E)$

elapsed time T : $A(\text{ellips}) = \pi ab$

Thus, according to Kepler's second law: $\dfrac{\frac{1}{2} ab (E - e \sin E)}{t - t_0} = \dfrac{\pi ab}{T}$ → $E - e \sin E = \frac{2\pi}{T}(t - t_0)$

Appendix D (see speed formula)

In a rectangular coordinate system, the speed of a planet V can be decomposed into two components, namely a horizontal component $V_x = \frac{dx}{dt}$ and a vertical component $V_y = \frac{dy}{dt}$. The following applies to the speed: $V^2 = V_x{}^2 + V_y{}^2 = (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2$

In a polar coordinate system, the velocity can be decomposed into a radial component $V_r = \frac{dr}{dt}$, which is directed along the planet-sun connection line, and a circular component $V_\theta = \frac{d\theta}{dt}$, which is perpendicular to it.

The relationship between V, V_r and V_θ can be derived using the formulas given in (5) for x and y:

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

Squaring the formulas (43), as time derivatives of the formulas (17), yields:

$$V_x = \frac{dx}{dt} = \cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt} \quad \rightarrow \quad (\frac{dx}{dt})^2 = \cos^2\theta(\frac{dr}{dt})^2 - 2r\sin\theta\cos\theta\frac{dr}{dt}\frac{d\theta}{dt} + r^2\sin^2\theta(\frac{d\theta}{dt})^2$$

$$V_y = \frac{dy}{dt} = \sin\theta\frac{dr}{dt} + r\cos\theta\frac{d\theta}{dt} \quad \rightarrow \quad (\frac{dy}{dt})^2 = \sin^2\theta(\frac{dr}{dt})^2 + 2r\sin\theta\cos\theta\frac{dr}{dt}\frac{d\theta}{dt} + r^2\cos^2\theta(\frac{d\theta}{dt})^2$$

Adding the previous two expressions gives, because of $\sin^2\theta + \cos^2\theta = 1$:

$$V^2 = (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = (\frac{dr}{dt})^2 + r^2(\frac{d\theta}{dt})^2 \qquad \text{expressed differently} \qquad V^2 = V_x{}^2 + V_y{}^2 = V_r{}^2 + r^2V_\theta{}^2$$

Appendix E (see Gaussian constant)

When an object in the solar system interacts exclusively with the sun, both masses constantly exert an equal but opposite force F on each other. This follows from Newton's third law: action = – reaction.

In addition, Newton's first law applies: $F = M\,a_M = -m\,a = G\dfrac{M\,m}{r^2}$. Where M is the mass of the sun, m the mass of the object, a_M the accelaration of the sun, a the accelaration of the object, r the mutual distance and G the gravitational constant. See also formule (52).

From this immediately follows the acceleration of the sun and of the object: $a_M = G\dfrac{m}{r^2}$ and $a = -G\dfrac{M}{r^2}$, where $a_M \ll a$.

By placing the sun in the origin of the coordinate system (focal point of the ellipse), the relative acceleration of the object is the difference of both, so: $a = -G\dfrac{M+m}{r^2}$. The minus sign indicates that the acceleration is directed toward the sun.

By comparing the acceleration with formula (49), namely $a = -\dfrac{4h^2}{p}\dfrac{1}{r^2}$ it immediately follows: $\dfrac{4h^2}{p} = G(M+m)$.

This is in accordance with the formulation of Kepler's third law in (51), where the mass of the object is negligible.

In the case of larger planets, where the mass is not negligible, the following applies to Kepler's third law instead of (51):

$$\frac{G(M+m)}{4\pi^2} = \frac{a^3}{T^2} \quad\rightarrow\quad GM\left(1+\frac{m}{M}\right) = 4\pi^2\,\frac{a^3}{T^2} \quad\rightarrow\quad \text{with } \mu = \frac{m}{M}: \quad \sqrt{GM} = \frac{2\pi}{\sqrt{1+\mu}}\cdot\frac{a^{3/2}}{T}$$

Carl Friedrich Gauss has symbolized the constant $\sqrt{GM}$, which is characteristic of our own solar system, with the symbol k.
In calculating this constant, he based himself on the then best available values of μ and T for the earth at its annual orbit around the sun.
The solar mass was used as the unit of mass ($M = 1$) and the sidereal day as the unit of time.

For the mass and the sidereal orbital period of the earth he took the values $\mu = \dfrac{1}{354710}$ respectively $T = 365.2563835$ days.

Gauss's gravitational constant can then be calculated: $k = \sqrt{GM} = \dfrac{2\pi}{\sqrt{1+\mu}}\cdot\dfrac{a^{3/2}}{T} = \dfrac{2\pi}{\sqrt{1+354710^{-1}}}\cdot\dfrac{1^{3/2}}{365.2563835} = 0.01720209895$

Over the next century, the mass and sidereal orbit of the Earth became more accurately known, but the International Astronomical Union decided to maintain the value of k as a fundamental reference value for a long time. The main reason was to avoid adjustment of the many already calculated orbital elements of planets, asteroids and comets, which would be brought about by changing the Gaussian constant.
With modern values of μ and T, the mean Earth-Sun distance now works out at 1.00000003 AU, a deviation in the eighth decimal place.
It was not until 2009 that the definition of k was abandoned, while the astronomical unit was redefined as: 1 AU = 1.495978707×10^{11} m.
The value $\sqrt{GM} = 0.017202098947$ is currently used instead of k.

Appendix F (see 4 equations)

Let: $\theta_2 - \theta_1 = 2f$ $\theta_2 + \theta_1 = 2F$ $E_2 - E_1 = 2g$ and $E_2 + E_1 = 2G$, then according to the formulas (29) holds for both positions:

$$\sqrt{r_1}\sin\frac{1}{2}\theta_1 = \sqrt{a(1+e)}\sin\frac{1}{2}E_1 \qquad \sqrt{r_1}\cos\frac{1}{2}\theta_1 = \sqrt{a(1-e)}\cos\frac{1}{2}E_1 \qquad \sqrt{r_2}\sin\frac{1}{2}\theta_2 = \sqrt{a(1+e)}\sin\frac{1}{2}E_2 \qquad \sqrt{r_2}\cos\frac{1}{2}\theta_2 = \sqrt{a(1-e)}\cos\frac{1}{2}E_2$$

Moreover, according to the formulas (32) holds for both positions: $E_1 - e\sin E_1 = \dfrac{k}{a^{3/2}}(t_1 - t_0)$ $\quad E_2 - e\sin E_2 = \dfrac{k}{a^{3/2}}(t_2 - t_0)$

Combining the formulas (29) for both positions yields:

$$\sqrt{r_2}\cos\frac{1}{2}\theta_2\cdot\sqrt{r_1}\cos\frac{1}{2}\theta_1 + \sqrt{r_2}\sin\frac{1}{2}\theta_2\cdot\sqrt{r_1}\sin\frac{1}{2}\theta_1 = \sqrt{a(1-e)}\cos\frac{1}{2}E_2\cdot\sqrt{a(1-e)}\cos\frac{1}{2}E_1 + \sqrt{a(1+e)}\sin\frac{1}{2}E_2\cdot\sqrt{a(1+e)}\sin\frac{1}{2}E_1$$

Left-hand side: $\sqrt{r_2}\cos\frac{1}{2}\theta_2\cdot\sqrt{r_1}\cos\frac{1}{2}\theta_1 + \sqrt{r_2}\sin\frac{1}{2}\theta_2\cdot\sqrt{r_1}\sin\frac{1}{2}\theta_1 = \sqrt{r_1 r_2}\cos\frac{1}{2}(\theta_2-\theta_1) = \sqrt{r_1 r_2}\cos f$

Right-hand side: $\sqrt{a(1-e)}\cos\frac{1}{2}E_2\cdot\sqrt{a(1-e)}\cos\frac{1}{2}E_1 + \sqrt{a(1+e)}\sin\frac{1}{2}E_2\cdot\sqrt{a(1+e)}\sin\frac{1}{2}E_1 = a(1-e)\cos\frac{1}{2}E_2\cos\frac{1}{2}E_1 + a(1+e)\sin\frac{1}{2}E_2\sin\frac{1}{2}E_1$

$$= a(\cos\frac{1}{2}E_2\cos\frac{1}{2}E_1 + \sin\frac{1}{2}E_2\sin\frac{1}{2}E_1) - ae(\cos\frac{1}{2}E_2\cos\frac{1}{2}E_1 - \sin\frac{1}{2}E_2\sin\frac{1}{2}E_1) = a\cos\frac{1}{2}(E_2-E_1) - ae\cos\frac{1}{2}(E_2+E_1) = a(\cos g - e\cos G)$$

$\rightarrow$ $\sqrt{r_1 r_2}\cos f = a(\cos g - e\cos G)$

$$\sqrt{r_2}\sin\frac{1}{2}\theta_2\cdot\sqrt{r_1}\cos\frac{1}{2}\theta_1 + \sqrt{r_2}\cos\frac{1}{2}\theta_2\cdot\sqrt{r_1}\sin\frac{1}{2}\theta_1 = \sqrt{a(1+e)}\sin\frac{1}{2}E_2\cdot\sqrt{a(1-e)}\cos\frac{1}{2}E_1 + \sqrt{a(1-e)}\cos\frac{1}{2}E_2\cdot\sqrt{a(1+e)}\sin\frac{1}{2}E_1$$

$$\sqrt{r_1 r_2}\sin\frac{1}{2}(\theta_2-\theta_1) = \sqrt{r_1 r_2}\sin f \qquad = a\sqrt{(1+e)(1-e)}\cdot(\sin\frac{1}{2}E_2\cos\frac{1}{2}E_1 - \cos\frac{1}{2}E_2\sin\frac{1}{2}E_1) = a\sqrt{(1+e)(1-e)}\sin g = \sqrt{ap}\sin g$$

$\rightarrow$ $\sqrt{r_1 r_2}\sin f = \sqrt{ap}\sin g$

Adding the formula (27) for both positions yields:

$$r_1 + r_2 = a(1-e\cos E_1) + a(1-e\cos E_2) = 2a - ae(\cos E_1 + \cos E_2) = 2a - 2ae\cos\frac{1}{2}(E_2-E_1)\cdot\cos\frac{1}{2}(E_2+E_1) = 2a - 2ae\cos g\cos G$$

The formula (32) gives for both positions after subtraction:

$$E_2 - e\sin E_2 - (E_1 - e\sin E_1) = E_2 - E_1 - e(\sin E_2 - \sin E_1) = E_2 - E_1 - 2e\sin\frac{1}{2}(E_2-E_1)\cdot\cos\frac{1}{2}(E_2+E_1) = 2g - 2e\sin g\cos G$$

$$\frac{k}{a^{3/2}}(t_2-t_0) - \frac{k}{a^{3/2}}(t_1-t_0) = \frac{k}{a^{3/2}}(t_2-t_1) = \frac{\tau}{a^{3/2}} \qquad \rightarrow \qquad \frac{\tau}{a^{3/2}} = 2g - 2e\sin g\cos G$$

Appendix G (see polynome X^{-1})

Suppose $Z(\xi) = \dfrac{1}{X(\xi)}$, then follows from this $\dfrac{dX}{d\xi} = -\dfrac{1}{Z^2} \cdot \dfrac{dZ}{d\xi}$. The differential equation of $Z(\xi)$ can be derived from the equation (74) by substituting X and $\dfrac{dX}{d\xi}$:

$$-2\xi(1-\xi) \cdot \frac{1}{Z^2(\xi)} \cdot \frac{dZ}{d\xi} + \frac{3}{Z(\xi)} \cdot (1-2\xi) - 4 = 0 \quad \rightarrow \quad 2\xi(1-\xi) \cdot \frac{dZ}{d\xi} + 6Z\xi - 3Z + 4Z^2 = 0$$

Suppose: $Z(\xi) = z_0 + z_1\xi + z_2\xi^2 + z_3\xi^3 + z_4\xi^4 + z_5\xi^5 + z_6\xi^6 + \ldots$ so $\dfrac{dZ}{d\xi} = z_1 + 2z_2\xi + 3z_3\xi^2 + 4z_4\xi^3 + 5z_5\xi^4 + 6z_6\xi^5 + \ldots$

Suppose: $Z^2(\xi) = y_0 + y_1\xi + y_2\xi^2 + y_3\xi^3 + y_4\xi^4 + y_5\xi^5 + y_6\xi^6 + \ldots = (z_0 + z_1\xi + z_2\xi^2 + z_3\xi^3 + z_4\xi^4 + z_5\xi^5 + z_6\xi^6 + \ldots)^2$

Expanding $Z^2(\xi)$ means for the terms y_n, expressed in z_n:

$y_0 = z_0^2$
$y_1 = 2z_0z_1$
$y_2 = 2z_0z_2 + z_1^2$
$y_3 = 2z_0z_3 + 2z_1z_2$
$y_4 = 2z_0z_4 + 2z_1z_3 + z_2^2$
$y_5 = 2z_0z_5 + 2z_1z_4 + 2z_2z_3$
$y_6 = 2z_0z_6 + 2z_1z_5 + 2z_2z_4 + z_3^2$
$y_7 = 2z_0z_7 + 2z_1z_6 + 2z_2z_5 + 2z_3z_4$
$y_8 = 2z_0z_8 + 2z_1z_7 + 2z_2z_6 + 2z_3z_5 + z_4^2$
$y_9 = 2z_0z_9 + 2z_1z_8 + 2z_2z_7 + 2z_3z_6 + 2z_4z_5$
$y_{10} = 2z_0z_{10} + 2z_1z_9 + 2z_2z_8 + 2z_3z_7 + 2z_4z_6 + z_5^2$
etcetera

So that for odd $n \geq 3$: $\quad y_n = 2\,z_0 \cdot z_n + 2\displaystyle\sum_{i=1}^{(n-1)/2} z_i \cdot z_{n-i}$ $\quad$ and for even $n \geq 4$: $\quad y_n = 2\,z_0 \cdot z_n + 2\displaystyle\sum_{i=1}^{n/2} z_i \cdot z_{n-i} + z_{n/2}^2$

Inserting the polynomials into the differential equation and arranging the terms with equal exponents gives:

$$+2\xi\frac{dZ}{d\xi} = \quad +2z_1\xi + 4z_2\xi^2 + 6z_3\xi^3 + 8z_4\xi^4 + 10z_5\xi^5 + \ldots$$

$$-2\xi^2\frac{dZ}{d\xi} = \quad -2z_1\xi^2 - 4z_2\xi^3 - 6z_3\xi^4 - 8z_4\xi^5 + \ldots$$

$$+6Z\xi = \quad +6z_0\xi + 6z_1\xi^2 + 6z_2\xi^3 + 6z_3\xi^4 + 6z_4\xi^5 + \ldots$$

$$-3Z = -3z_0 - 3z_1\xi - 3z_2\xi^2 - 3z_3\xi^3 - 3z_4\xi^4 - 3z_5\xi^5 + \ldots$$

$$+4Z^2 = +4z_0^2 + 8z_0z_1\xi + 4(2z_0z_2 + z_1^2)\xi^2 + 4(2z_0z_3 + 2z_1z_2)\xi^3 + 4(2z_0z_4 + 2z_1z_3 + z_2^2)\xi^4 + 4(2z_0z_5 + 2z_1z_4 + 2z_2z_3)\xi^5 + \ldots = 0$$

The first 3 terms of $Z(\xi) = \dfrac{1}{X(\xi)}$ are calculated manually from the above data:

$$-3z_0 \quad +4y_0 = -3z_0 + 4z_0^2 \quad = -3z_0 + \; 4z_0^2 = 0 \quad \rightarrow \quad z_0 = \frac{3}{4} \text{ (substitute: } 8z_0 = 6\text{)}$$

$$+6z_0 - 1z_1 + 4y_1 = +6z_0 - 1z_1 + 8z_0z_1 \quad = +6z_0 + \; 5z_1 \; = 0 \quad \rightarrow \quad z_1 = \frac{2}{5}(-3z_0) = -\frac{9}{10}$$

$$+4z_1 + 1z_2 + 4y_2 = +4z_1 + 1z_2 + 8z_0z_2 + 4z_1^2 \quad = +4z_1 + \; 7z_2 + 4z_1^2 = 0 \quad \rightarrow \quad z_2 = \frac{2}{7}(-2z_1 - 2z_1^2) = \frac{9}{175}$$

$$+2z_2 + 3z_3 + 4y_3 = +2z_2 + 3z_3 + 8z_0z_3 + 8z_1z_2 \quad = +2z_2 + \; 9z_3 + 8z_1z_2 = 0 \quad \rightarrow \quad z_3 = \frac{2}{9}(-1z_2 - 4z_1z_2)$$

$$0z_3 + 5z_4 + 4y_4 = \; 0z_3 + 5z_4 + 8z_0z_4 + 8z_1z_3 + 4z_2^2 \quad = \; 0z_3 + 11z_4 + 8z_1z_3 + 4z_2^2 = 0 \quad \rightarrow \quad z_4 = \frac{2}{11}(\;0z_3 - 4z_1z_3 - 2z_2^2)$$

$$-2z_4 + 7z_5 + 4y_5 = -2z_4 + 7z_5 + 8z_0z_5 + 8z_1z_4 + 8z_2z_3 \quad = -2z_4 + 13z_5 + 8(z_1z_4 + z_2z_3) = 0 \quad \rightarrow \quad z_5 = \frac{2}{13}(+1z_4 - 4z_1z_4 - 4z_2z_3)$$

$$-4z_5 + 9z_6 + 4y_6 = -4z_5 + 9z_6 + 8z_0z_6 + 8z_1z_5 + 8z_2z_4 + 2z_3^2 = -4z_5 + 15z_6 + 8(z_1z_5 + z_2z_4) + 4z_3^2 = 0 \quad \text{etcetera}$$

When $n \geq 3$:

For odd n: $$z_n = \frac{2}{2n+3}\left[(n-4)\cdot z_{n-1} - 4\sum_{i=1}^{(n-1)/2} z_i \cdot z_{n-i}\right]$$

for even n: $$z_n = \frac{2}{2n+3}\left[(n-4)\cdot z_{n-1} - 2z_{n/2}^2 - 4\sum_{i=1}^{n/2-1} z_i \cdot z_{n-i}\right]$$

Elaboration of the above formulas for the polynomials $Z(\xi) = \dfrac{1}{X(\xi)}$ and $\nu(\xi)$ for the first 14 terms:

$$\frac{1}{X(\xi)} = z_0 + z_1\xi + z_2\xi^2 + z_3\xi^3 + z_4\xi^4 + z_5\xi^5 + \ \ldots$$

z_0	$3/4$
z_1	$-9/10$
z_2	$9/175$
z_3	$26/875$
z_4	$6228/336875$
z_5	$265896/21896875$
z_6	$19139024/2299171875$
z_7	$385073504/65143203125$
z_8	$2060869592128/476522530859375$
z_9	$9945695595904/3063359126953125$
z_{10}	$186453795500433152/748021242816503906250$
z_{11}	$3492931512980992/17895245043457031 25$
z_{12}	$8763849223728870707 2/5634551318172143554687 5$
z_{13}	$262831910980243925463 04/20879142940115665283203125$

$$\nu(\xi) = c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + c_5\xi^5 + \ \ldots$$

c_0	0
c_1	0
c_2	$2/35$
c_3	$52/1575$
c_4	$1384/67375$
c_5	$59088/4379375$
c_6	$38278048/4138509375$
c_7	$770147008/117257765625$
c_8	$4121739184256/857740555546875$
c_9	$19891391191808/5514046428515625$
c_{10}	$372907591000866304/1346438237069707031 25$
c_{11}	$6985863025961984/3221144107822265625$
c_{12}	$175276984474577414144/10142192372709858398437 5$
c_{13}	$5256638219604878509260 8/37582457292208197509765625$

Given $z_0 = \dfrac{3}{4}$, $z_1 = -\dfrac{6}{5}z_0$ and $z_2 = -\dfrac{4}{7}z_1(1+z_1)$, the coefficients of $\dfrac{1}{X(\xi)}$ can be generated as follows:

at odd $n \geq 3$: $$z_n = \frac{2}{2n+3}\left[(n-4)\cdot z_{n-1} - 4\sum_{i=1}^{(n-1)/2} z_i \cdot z_{n-i}\right]$$

at even $n \geq 4$: $$z_n = \frac{2}{2n+3}\left[(n-4)\cdot z_{n-1} - 2z_{n/2}{}^2 - 4\sum_{i=1}^{n/2-1} z_i \cdot z_{n-i}\right]$$

The relation between the coefficients c_n and z_n is:

$$c_0 = \frac{10}{9}z_0 - \frac{5}{6} = 0 \qquad c_1 = \frac{10}{9}\cdot z_1 + 1 = 0$$

for $n \geq 2$: $$c_n = \frac{10}{9}\cdot z_n$$

Appendix H (see limit X for $\xi \to 0$)

Here l'Hôpital's rule applies: if 2 differentiable functions $f(x)$ and $g(x)$ satisfy the condition $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 0$, then for the quotient of these functions in relation to the first derivatives the following holds: $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$

For the case to be proven: $\lim_{\xi\to 0} X(\xi) = \lim_{g\to 0} \frac{2g - \sin 2g}{\sin^3 g} = \lim_{g\to 0} \frac{2 - 2\cos 2g}{3\sin^2 g \cos g} = \lim_{g\to 0} \frac{2 - 2(1 - 2\sin^2 g)}{3\sin^2 g \cos g} = \lim_{g\to 0} \frac{4\sin^2 g}{3\sin^2 g \cos g} = \lim_{g\to 0} \frac{4}{3\cos g} = \frac{4}{3}$

Appendix I (see functions η_a and η_b)

An ellipse is a concave, closed curve, whose area ratio of sector and triangle η is always greater than 1.

This immediately follows from (65), (68), (73) and (81) with: $\eta = 1 + X(\xi)(\lambda + \xi)$ and $\lambda > 0$, $\xi > 0$, $X(\xi) > \frac{4}{3}$, $\eta > 1$.

Given: $\eta_a(\xi) = 1 + X(\xi)(\lambda + \xi) \quad \to \quad \eta_a(0) = 1 + X(0)\cdot\lambda = 1 + \frac{4}{3}\lambda$ and $\eta_b(\xi) = \sqrt{\frac{\mu}{\lambda + \xi}} \quad \to \quad \eta_b(0) = \sqrt{\frac{\mu}{\lambda}}$

To be proven: $\eta_b(0) > \eta_a(0)$

Proof: If $\eta_b(0) > \eta_a(0)$, then $\eta_b{}^2(0) > \eta_a{}^2(0)$ and must hold: $\frac{\mu}{\lambda} > (1 + \frac{4}{3}\lambda)^2 \quad \to \quad \mu > \lambda\cdot(1 + \frac{4}{3}\lambda)^2$

Suppose that (ξ_0, η_0) is the solution of the equation $\eta_a(\xi) = \eta_b(\xi)$, where η_0 is the area ratio sought.

Then according to (68) and (69): $\eta_0{}^2 = \frac{\mu}{\lambda + \xi_0}$ and $\eta_0 = 1 + X(\xi_0)(\lambda + \xi_0) \quad \to \quad \mu = \frac{(\eta_0 - 1)\cdot\eta_0{}^2}{X(\xi_0)}$ and $\lambda = \frac{\eta_0 - 1}{X(\xi_0)} - \xi_0$

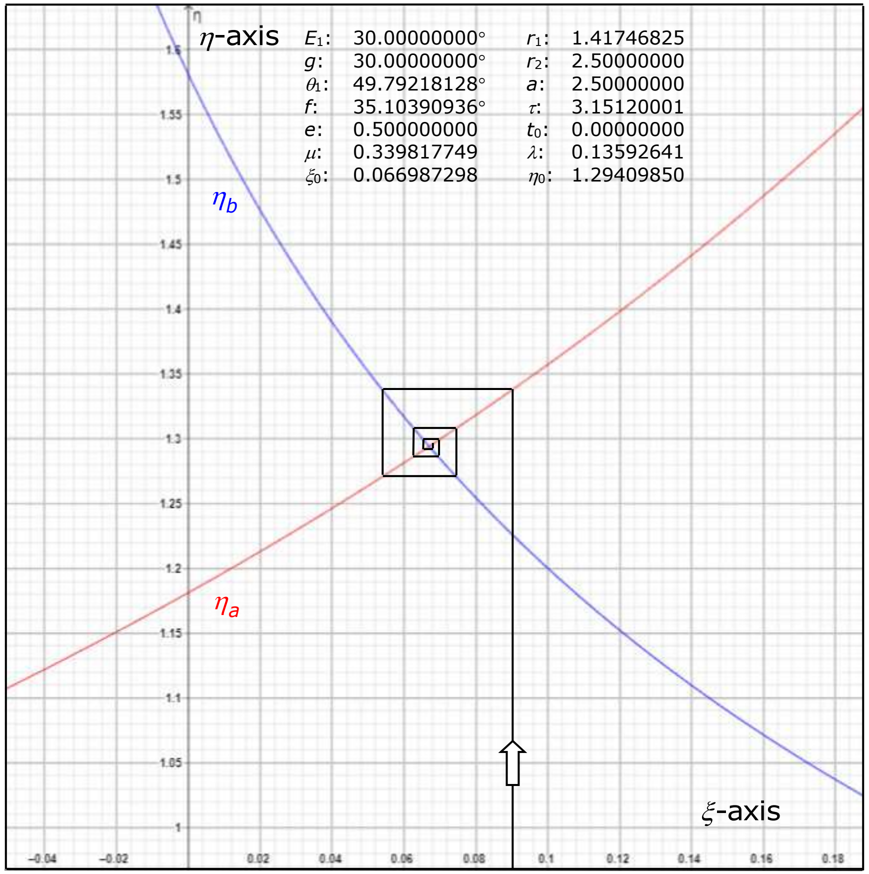
So to prove: $\frac{\eta_0 - 1}{X(\xi_0)}\cdot\eta_0{}^2 > (\frac{\eta_0 - 1}{X(\xi_0)} - \xi_0)\cdot(1 + \frac{4}{3}(\frac{\eta_0 - 1}{X(\xi_0)} - \xi_0))^2$ Immediately it can be seen that: $\frac{\eta_0 - 1}{X(\xi_0)} > \frac{\eta_0 - 1}{X(\xi_0)} - \xi_0$,

so that it is yet to be proved: $\eta_0 > 1 + \frac{4}{3}(\frac{\eta_0 - 1}{X(\xi_0)} - \xi_0)$ Suppose: $\alpha = \frac{4}{3}\cdot\frac{1}{X(\xi_0)}$ with $0 < \alpha < 1$, because $X(\xi_0) > \frac{4}{3}$

$\eta_0 > 1 + \alpha\eta_0 - \alpha - \frac{4}{3}\xi_0 \quad \to \quad \eta_0(1 - \alpha) > 1 - \alpha - \frac{4}{3}\xi_0 \quad \to \quad \eta_0 > 1 - \frac{4}{3}\cdot\frac{\xi_0}{1 - \alpha}$ but $1 - \frac{4}{3}\cdot\frac{\xi_0}{1 - \alpha} < 1$ q.e.d.

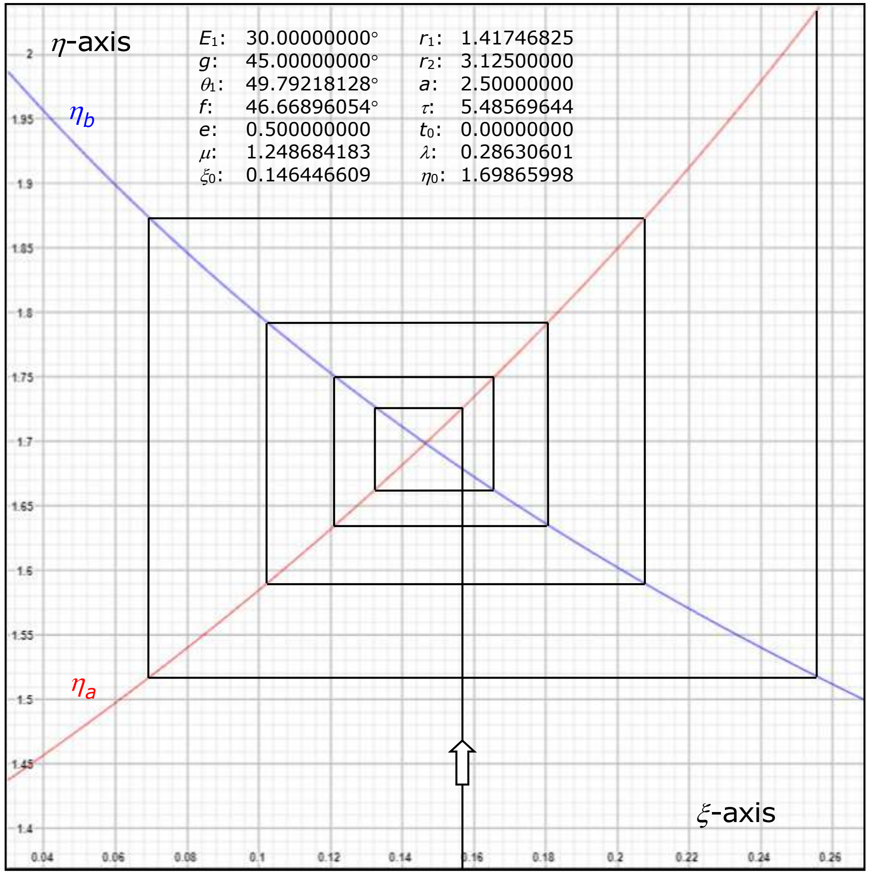
So for all solutions with (ξ_0, η_0) with $0 < \xi_0 < \frac{1}{2}$ and $1 < \eta_0 < \infty$ holds: $\eta_b(0) > \eta_a(0)$.

Appendix J (see divergence on iteration)



The iteration started with $\xi = \sin^2(\frac{1}{2}f) = 0.0909$.

There is quite good convergence.



The iteration started with $\xi = \sin^2(\frac{1}{2}f) = 0.1569$.

There is divergence.

Appendix K (see limiting case ellipse)

The parabola is the limiting case of the ellipse, where the eccentricity $e=1$. So the orbital equation of the parabola is: $r=\frac{p}{1+\cos\theta}$

According to (10) $a=\frac{p}{1-e^2}$, so that the semimajor axis a no longer has any real meaning here, after all $\lim_{e\to1} a=\lim_{e\to1}\frac{p}{1-e^2}=\infty$.

Usually the perihelion distance q is used as a parameter for parabolic orbits. When $\theta=0$ holds that: $q=\frac{p}{1+\cos 0}=\frac{p}{2}$.

In order not to have to introduce a new symbol, however, the parameter p will continue to be used here.

For a parabola, the factor g as the difference of 2 eccentric anomalies equals 0. This also means $\xi=\sin^2\frac{1}{2}g=0$.

This becomes clear by squaring equation (59): $r_1r_2\sin^2 f=ap\sin^2 g=\frac{p^2}{1-e^2}\sin^2 g=\frac{2p^2}{1-e^2}\sin^2\frac{1}{2}g\cos^2\frac{1}{2}g=\frac{2p^2}{1-e^2}\cdot\xi(1-\xi)$

From this follows: $\xi(1-\xi)=\frac{1-e^2}{2p^2}\cdot r_1r_2\sin^2 f$, hence $\lim_{e\to1}\xi(1-\xi)=\lim_{e\to1}\frac{1-e^2}{2p^2}\cdot r_1r_2\sin^2 f=0 \quad\rightarrow\quad \xi(1-\xi)=0$ so $\xi=0$

According to (70), (74) and (75) then applies: $X(\xi=0)=\frac{4}{3}$ and $\eta=1+\frac{4}{3}\lambda=1+\frac{4}{3}(\frac{r_1+r_2}{2\kappa}-\frac{1}{2})=\frac{1}{3}(1+2\cdot\frac{r_1+r_2}{\kappa})$

The following is derived for the parameter κ at (93): $\kappa=2\sqrt{r_1r_2}\cos f=\sqrt{2(r_1r_2+x_1x_2+y_1y_2)}$

The sector/triangle area ratio η can also be derived directly from the above equation of the parabola (see figure 8). The 'detour' via the eccentric anomaly is not necessary here and the sector area can be expressed directly in the true anomaly:

$$A_{\text{sector}}=\frac{1}{2}\int_{\theta_1}^{\theta_2}r^2\mathrm{d}\theta=\frac{1}{2}\int_{\theta_1}^{\theta_2}(\frac{p}{1+\cos\theta})^2\mathrm{d}\theta=\frac{p^2}{8}\int_{\theta_1}^{\theta_2}\frac{\mathrm{d}\theta}{\cos^4\frac{1}{2}\theta}=\frac{p^2}{4}\int_{\theta_1}^{\theta_2}\frac{\mathrm{d}(\frac{1}{2}\theta)}{\cos^4\frac{1}{2}\theta}=\frac{p^2}{4}\int_{\theta_1}^{\theta_2}\frac{\mathrm{d}(\tan\frac{1}{2}\theta)}{\cos^2\frac{1}{2}\theta}=\frac{p^2}{4}\int_{\theta_1}^{\theta_2}(1+\tan^2\frac{1}{2}\theta)\,\mathrm{d}(\tan\frac{1}{2}\theta)$$

$$A_{\text{sector}}=\frac{p^2}{4}(\tan\frac{1}{2}\theta+\frac{1}{3}\tan^3\frac{1}{2}\theta)\Big|_{\theta_1}^{\theta_2}=\frac{p^2}{12}\left[3(\tan\frac{1}{2}\theta_2-\tan\frac{1}{2}\theta_1)+(\tan^3\frac{1}{2}\theta_2-\tan^3\frac{1}{2}\theta_1)\right]$$

$$A_{\text{triangle}}=\frac{1}{2}\cdot\text{base}\cdot\text{height}=\frac{1}{2}r_1r_2\sin 2f=\frac{1}{2}r_1r_2\sin(\theta_2-\theta_1)=\frac{1}{2}r_1r_2(\sin\theta_2\cos\theta_1-\sin\theta_1\cos\theta_2)$$

See also derivation of formule (57).

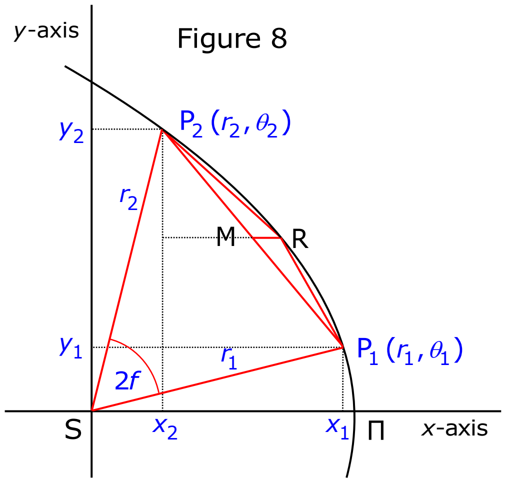
The great mathematician Archimedes (287-212 BC) already knew how to determine the area of a parabolic segment. He used the so-called exhaustion method, a predecessor of integral calculus. To determine η, rectangular coordinates will be used.

After all, according to (5): $x = r\cos\theta$, $y = r\sin\theta$ and $r^2 = r^2\cos^2\theta + r^2\sin^2\theta = x^2 + y^2$.

From the orbit equation follows: $p = r + r\cos\theta = r + x$, and the equation of the parabola is thus:

$$y^2 = r^2 - x^2 = (p - x)^2 - x^2 \qquad \rightarrow \qquad y^2 = p^2 - 2px$$

In addition, the following applies: $\tan\frac{1}{2}\theta = \frac{\sin\theta}{1+\cos\theta} = \frac{r\sin\theta}{r + r\cos\theta} = \frac{y}{p}$



Application of the previous relation to the sector area obtained via integral calculus then gives:

$$A_{\text{sector}} = \frac{p^2}{12}\left[3(\tan\frac{1}{2}\theta_2 - \tan\frac{1}{2}\theta_1) + (\tan^3\frac{1}{2}\theta_2 - \tan^3\frac{1}{2}\theta_1)\right] = \frac{p^2}{12}\left[3\cdot\frac{y_2 - y_1}{p} + \frac{y_2{}^3 - y_1{}^3}{p^3}\right] =$$

$$A_{\text{sector}} = \frac{p^2}{12}\left[3\cdot\frac{y_2 - y_1}{p} + \frac{(y_2 - y_1)(y_2{}^2 + y_1 y_2 + y_1{}^2)}{p^3}\right] = \frac{y_2 - y_1}{12p}(3p^2 + y_2{}^2 + y_1 y_2 + y_1{}^2)$$

In addition, for the area of the triangle:

$$A_{\text{triangle}} = \frac{1}{2}r_1 r_2 \sin 2f = \frac{1}{2}r_1 r_2(\sin\theta_2\cos\theta_1 - \sin\theta_1\cos\theta_2) = \frac{1}{2}(r_2\sin\theta_2\cdot r_1\cos\theta_1 - r_1\sin\theta_1\cdot r_2\cos\theta_2) = \frac{1}{2}(y_2 x_1 - y_1 x_2)$$

By expressing x in y by the orbit equation with $x = \frac{p^2 - y^2}{2p}$, x can be eliminated, such that:

$$A_{\text{triangle}} = \frac{1}{2}(y_2 x_1 - y_1 x_2) = \frac{1}{2}(y_2\frac{p^2 - y_1{}^2}{2p} - y_1\frac{p^2 - y_2{}^2}{2p}) = \frac{1}{4p}(p^2 y_2 - y_1{}^2 y_2 - p^2 y_1 + y_2{}^2 y_1) = \frac{1}{4p}(y_2 - y_1)(y_1 y_2 + p^2)$$

So the area ratio sector/triangle for the parabola can be written as:

$$\eta = \frac{A_{\text{sector}}}{A_{\text{triangle}}} = \frac{\frac{y_2 - y_1}{12p}(3p^2 + y_2{}^2 + y_1 y_2 + y_1{}^2)}{\frac{1}{4p}(y_2 - y_1)(y_1 y_2 + p^2)} = \frac{1}{3}\cdot\frac{3p^2 + 3y_1 y_2 + y_2{}^2 - 2y_1 y_2 + y_1{}^2}{y_1 y_2 + p^2} = \frac{1}{3}\cdot(3 + \frac{y_2{}^2 - 2y_1 y_2 + y_1{}^2}{y_1 y_2 + p^2}) = 1 + \frac{1}{3}\cdot\frac{(y_2 - y_1)^2}{y_1 y_2 + p^2}$$

Since $A_{segment} = A_{\text{sector}} - A_{\text{triangle}}$ follows for the parabolic segment P_1P_2R:

$$A_{segment} = \frac{y_2 - y_1}{12p}(3p^2 + y_2{}^2 + y_1 y_2 + y_1{}^2) - \frac{1}{4p}(y_2 - y_1)(y_1 y_2 + p^2) = \frac{y_2 - y_1}{12p}(3p^2 + y_2{}^2 + y_1 y_2 + y_1{}^2 - 3y_1 y_2 - 3p^2) = \frac{(y_2 - y_1)^3}{12p}$$

Archimedes had the parabola intersected by the chord P_1P_2, took the center M of the chord and bisected both the parabolic segment P_1P_2R and the triangle P_1P_2R with the horizontal line segment MR. He then proved that the areas of both are related as 4:3.

Ergo: $$A_{segment} = \frac{4}{3}\cdot A_{\Delta P_1P_2R} = \frac{4}{3}\cdot(A_{\Delta P_1MR} + A_{\Delta P_2MR}) = \frac{4}{3}\cdot 2\cdot(\frac{1}{2}\cdot\text{base}\cdot\text{height}) = \frac{4}{3}\cdot\text{MR}\cdot\frac{y_2 - y_1}{2} = \frac{2}{3}\cdot\text{MR}\cdot(y_2 - y_1)$$

So it follows from this: $A_{segment} = \frac{(y_2 - y_1)^3}{12p} = \frac{2}{3} \cdot \mathrm{MR} \cdot (y_2 - y_1) \quad \rightarrow \quad \mathrm{MR} = \frac{(y_2 - y_1)^2}{8p}$

This can be verified by means of the x-coordinates of the points M (the midpoint of chord P_1P_2) and R on the parabola:

$$x_M = \frac{x_1 + x_2}{2} = \frac{p^2 - y_1^2}{4p} + \frac{p^2 - y_2^2}{4p} = \frac{2p^2 - y_1^2 - y_2^2}{4p} \quad \text{and} \quad x_R = \frac{p^2 - y_R^2}{2p} = \frac{p^2 - y_M^2}{2p} = \frac{p^2 - \frac{1}{4}(y_1 + y_2)^2}{2p} = \frac{4p^2 - (y_1 + y_2)^2}{8p}$$

$$x_R - x_M = \frac{4p^2 - (y_1 + y_2)^2}{8p} - \frac{2p^2 - y_1^2 - y_2^2}{4p} = \frac{4p^2 - y_1^2 - 2y_1y_2 - y_2^2 - 4p^2 + 2y_1^2 + 2y_2^2}{8p} = \frac{y_1^2 - 2y_1y_2 + y_2^2}{8p} = \frac{(y_2 - y_1)^2}{8p} = \mathrm{MR}$$

The point R forms the tangent point of the tangent to the parabola, parallel to the chord P_1P_2. This follows directly from the orbital equation:

$$2y_R dy = -2p dx \quad \rightarrow \quad (\frac{dy}{dx})_R = -\frac{p}{y_R} = -\frac{2p}{y_2 + y_1} = -\frac{2p}{y_2 + y_1} \cdot \frac{y_2 - y_1}{y_2 - y_1} = \frac{-2p(y_2 - y_1)}{y_2^2 - y_1^2} = \frac{-2p(y_2 - y_1)}{p^2 - 2px_2 - p^2 + 2px_1} = \frac{y_2 - y_1}{x_2 - x_1} = (\frac{dy}{dx})_M$$

So far, 2 equations have been derived for the sector/triangle area ratio for the parabola:

- via Gauss's theory as the limiting case of the ellipse with eccentricity $e = 1$: $\quad \eta = 1 + \frac{4}{3}\lambda = \frac{1}{3}(1 + 2 \cdot \frac{r_1 + r_2}{\kappa})$
- by integrating the orbital equation and the geometry of the parabola: $\quad \eta = 1 + \frac{1}{3} \cdot \frac{(y_2 - y_1)^2}{y_1y_2 + p^2}$

According to (93) and (94) also holds: $\kappa^2 = 4r_1r_2\cos^2 f = 2(r_1r_2 + x_1x_2 + y_1y_2)$ and $\lambda = \frac{r_1 + r_2}{2\kappa} - \frac{1}{2}$

That the two expressions given above for η are equivalent will now be shown.

The following relation is used for this: $\kappa = \frac{y_1y_2}{p} + p$

Proof: $\kappa^2 = (\frac{y_1y_2}{p} + p)^2 = \frac{y_1^2}{p} \cdot \frac{y_2^2}{p} + 2y_1y_2 + p^2$ According to the orbital equation: $p = r + x$ and $\frac{y^2}{p} = p - 2x = (r + x) - 2x = r - x$

Subsitution of $\frac{y_1^2}{p} = r_1 - x_1$, $\frac{y_2^2}{p} = r_2 - x_2$ and $p = r_1 + x_1 = r_2 + x_2$ then gives:

$$\kappa^2 = \frac{y_1^2}{p} \cdot \frac{y_2^2}{p} + 2y_1y_2 + p^2 = (r_1 - x_1)(r_2 - x_2) + 2y_1y_2 + (r_1 + x_1)(r_2 + x_2) = 2r_1r_2 + 2x_1x_2 + 2y_1y_2 \qquad \text{q.e.d.}$$

Finally, λ will be expressed in y_1, y_2 and p via the orbital equation:

$$\lambda = \frac{r_1 + r_2}{2\kappa} - \frac{1}{2} = \frac{p - x_1 + p - x_2}{2(\frac{y_1 y_2}{p} + p)} - \frac{1}{2}) = \frac{1}{2}(\frac{p^2 - 2px_1 + p^2 - 2px_2 + 2p^2}{2y_1 y_2 + 2p^2} - 1) = \frac{1}{2}(\frac{y_1{}^2 + y_2{}^2 + 2p^2}{2y_1 y_2 + 2p^2} - 1) = \frac{1}{2}(\frac{y_1{}^2 + y_2{}^2 + 2p^2 - 2y_1 y_2 - 2p^2}{2y_1 y_2 + 2p^2})$$

$$\lambda = \frac{1}{2} \cdot \frac{y_1{}^2 - 2y_1 y_2 + y_2{}^2}{2y_1 y_2 + 2p^2} = \frac{1}{4} \cdot \frac{(y_1 - y_2)^2}{y_1 y_2 + p^2}$$

This proves that for the area ratio sector/triangle η at the parabola holds:

$$\eta = 1 + \frac{4}{3}\lambda = 1 + \frac{4}{3} \cdot \frac{1}{4} \cdot \frac{(y_1 - y_2)^2}{y_1 y_2 + p^2} = 1 + \frac{1}{3} \cdot \frac{(y_2 - y_1)^2}{y_1 y_2 + p^2} \qquad \text{ánd} \qquad \eta = 1 + \frac{4}{3}\lambda = 1 + \frac{4}{3}(\frac{r_1 + r_2}{2\kappa} - \frac{1}{2}) = \frac{1}{3}(1 + 2 \cdot \frac{r_1 + r_2}{\kappa})$$

Appendix L (see calculation eccentric anomaly)

Newton's method is based on the approximation of a function in the vicinity of the zero point using a Taylor series:

$$f(x) = f(a) + \frac{df(a)}{dx}(x - a) + \frac{1}{2!} \cdot \frac{d^2 f(a)}{dx^2}(x - a)^2 + \frac{1}{3!} \cdot \frac{d^3 f(a)}{dx^3}(x - a)^3 + \ \dots \ = 0$$

Here a is an estimate of the x-value to be calculated for which $f(x) = 0$.

If only the first derivative is used and $x - a$ is replaced by $x_n - x_{n-1}$, then with iterative approximations:

$$f(x_n) = f(x_{n-1}) + \frac{df(x_{n-1})}{dx}(x_n - x_{n-1}) = 0 \quad \rightarrow \quad x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad \text{where} \quad \frac{df(x_{n-1})}{dx} \quad \text{is denoted as} \quad f'(x_{n-1}).$$

The formula derived above is the mathematical formulation of Newton's method.

In this specific case, the value to be approximated is the eccentric anomaly E, to be solved from (32): $f(E) = E - e\sin E - M = 0$.

The variable x is replaced by the variable E, so that the formulation is as follows: $E_n = E_{n-1} - \dfrac{f(E_{n-1})}{f'(E_{n-1})}$

In the last equation then holds: $f(E_{n-1}) = E_{n-1} - e\sin E_{n-1} - M = 0$ and for the first derivative: $f'(E_{n-1}) = 1 - e\cos E_{n-1}$.

Filling in these expressions gives the formula listed on page 9: $E_n = E_{n-1} + \dfrac{M + e\sin E_{n-1} - E_{n-1}}{1 - e\cos E_{n-1}}$